Classifying Logics: Abstract Model Theory

An Introduction

William J Angus

"I certify that this project report has been written by me, is a record of work carried out by me, and is essentially different from work undertaken for any other purpose or assessment."

William James Angus

Abstract

Abstract Model Theory is a field of Mathematical Logic, in which Mathematicians study the relations between various "Logics" – which are methods of defining "truth in a structure" (that is, what holds in a Group, or an Ordering, or a Graph, &c.). This project provides an introduction to the field, with no prerequisites in Mathematical Logic. We shall encounter various properties of Logics, such as Compactness, and the Löwenheim-Skolem-Tarski Property; and we shall meet different kinds of Logics: First-Order Logic, Boolean Logic, and Infinitary Logic, which we shall compare. The project ends with a foundational result in Abstract Model Theory: Lindström's Theorem, which neatly characterises First-Order Logic by its properties. This is, to the best of my knowledge, the first self-contained introduction to Lindström's Theorem, which we work up to, by proving results about "Orthodox Logics".

Contents

1 Introduction

I intend this project to serve as an introduction to the field of Abstract Model Theory, which is a field of Mathematical Logic, in which researchers seek to categorise and compare different Logics. And, in particular, I believe that this will be the first comprehensive introduction to the field, from no pre-requisites in Mathematical Logic, up to Lindström's Theorem – and, so I hope that this project will serve as useful to those whom want to understand such a key result. In order to get the most out of reading this introduction, the reader should have some level of mathematical maturity, for example being a 4th year student at the University of St Andrews, and should been very familiar with basic algebra (for example, this familiarity could be picked up by the reader having done four of MT2501, MT2505, MT3501, MT3505, and MT4003); in addition, it would be useful if the reader had some knowledge of the hierarchy of infinities (i.e., know the difference between countable and uncountable infinities).

In order to accommodate such 'lax' pre-requisites, along the way, we shall encounter, and be introduced to different subfields of Mathematical Logic, namely Model Theory and Set Theory.

We shall begin, in the next section, by introducing the concepts of Boolean Algebra, Model Theory, and Logic (in our specific sense), and we shall prove that Boolean Algebras are Logics. Then, in Section 3, we shall meet basic Set Theory, and encounter Infinitary Logics – an extension of First-Order Logic, allowing infinitely long formulæ. Next, in Section 4, we will meet and prove some properties of First-Order Logic, comparing them to those of the other Logics we have met along the way. Following this, in Section 5, we will be introduced to – and prove some results about – the idea of "expressibility": the properties that a Logic can "describe". Finally, in Section 6, we will meet and prove Lindström's Theorem – a neat result that completely characterises First-Order Logic, and the foundational result in Abstract Model Theory.

Before we begin, however, here is a list of the notation that shall be assumed:

- If X and Y are sets, then $X \subseteq Y$ means "improper subset"; that is, $X \subseteq X$.
- If X is a set, $\mathcal{P}(X)$ represents the powerset of X.
- If f is a function from a set X to a set Y, we write $f : X \to Y$.
- Given sets X and Y, we write $X \times Y$ to denote the corresponding Cartesian product.
- The empty set is written as a stylised version of the Nordic letter 'ø': $\emptyset$.
- If A and B are sets, we write $f : A \to B[a \mapsto g(a)]$ to mean that f is a function from A to B, and is defined such that each a in A is taken to $g(a)$.
- We write $A := B$ to mean 'A is defined to be B'.
- If X is a set, we write $|X|$ to denote the size, or cardinality, of X.
- $\mathbb{N}$ contains 0.
- "Countable" means "not uncountable"; i.e., we regard finite (including empty) sets as "countable".
- If a proof ends with '■', then I am claiming to have "filled in the details" (and so, the square), by having done a substantial amount of work (this is not to say that the proof is novel, or that the claim is novel, but that I have not copied the proof, or have done significant work to fill in

the details of a proof that appears somewhere else); if however, a proof ends with '□', then I regard the proof as being "by the book", and claim no originality.

2 Boolean Algebras, Models, and Logics

In this section, we introduce three notions: Boolean Algebras, Model Theory, and Logics. We begin by defining and providing some preliminary results about "Boolean Algebras". This will be reminiscent of the Algebra modules at the University of St Andrews. Then, we will introduce the basic tenants of Model Theory, using Boolean Algebras as examples. Next, we will see what a "Logic" is, and we will see our first example of a Logic: Boolean Algebras; and finally, following this, we shall see that First-Order Logic is indeed a "Logic" under our definition.

2.1 Boolean Algebras

First, we recall some basic definitions about "operations", which shall be used to define "Boolean Algebra".

Definition 2.1.1 (Unary Operation). Let X be a non-empty set. If $f : X \to X$ is a function, we say that f *is a unary operation on* X.

Definition 2.1.2 (Binary Operation). Let X be a non-empty set. If $f : X \times X \to X$ is a function, we say that f *is a binary operation on* X.

We are now ready to define "Boolean Algebra", following the axiomatisation given in [GH09]. This will be done in the same manner as Groups, Rings, &c., by giving a list of axioms that the operations on the underlying set must obey.

Definition 2.1.3 (Boolean Algebra). Let B be an arbitrary non-empty set. We say B is a *Boolean Algebra* if and only if it has two binary operations $\wedge$ (read 'meet') and $\vee$ (read 'join'), a unary operation $\neg$ (read 'complement'), and two constants (distinguished elements of B) $\top$ (read 'top') and $\bot$ (read 'bottom') that satisfy the following list of axioms for all x, y, and z in B:

Commutativity $x \wedge y = y \wedge x$; and $x \vee y = y \vee x$;

Distributivity $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$; and $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$;

Identity $x \wedge \top = x$; and $x \vee \bot = x$;

Complements $x \wedge \neg x = \bot$; and $x \vee \neg x = \top$;

Associativity $x \wedge (y \wedge z) = (x \wedge y) \wedge z$; and $x \vee (y \vee z) = (x \vee y) \vee z$;

De Morgan $\neg(x \wedge y) = \neg x \vee \neg y$; and $\neg(x \vee y) = \neg x \wedge \neg y$;

Extremes $x \wedge \bot = \bot$; and $x \vee \top = \top$;

Invariance $x \wedge x = x$; and $x \vee x = x$;

Duality $\neg\bot = \top$; and $\neg\top = \bot$;

Double Complement Elimination $\neg\neg x = x$.

This list of axioms is, in fact, excessive. The first four are enough to axiomatise the concept of "Boolean Algebra". That is to say that the other axioms all follow from the first four. However, it useful to give a much bigger list to better understand the structure of Boolean Algebras. We shall now see, but not prove, a powerful fact about (finite) Boolean Algebras:

Theorem 2.1.4. *Let B be a finite Boolean Algebra. Then, there exists a finite set X such that B is isomorphic to $\mathcal{P}(X)$, where the meet operation $\wedge$ corresponds to the intersection operation $\cap$ defined on sets in $\mathcal{P}(X)$; the join operation $\vee$ corresponds to the union operation $\cup$ defined on sets in $\mathcal{P}(X)$; the complement operation $\neg$ corresponds to the set complement operation $\cdot^{\complement}$ defined on sets in $\mathcal{P}(X)$ so that $A \mapsto \mathcal{P}(X) \setminus A$; the top element $\top$ corresponds to the set X; and the bottom element $\bot$ corresponds to the empty set $\emptyset$.*

Moreover, given any finite set X, there is some Boolean Algebra B isomorphic to $\mathcal{P}(X)$, with respect to the same operations above.

Proof: see [GH09, p. 127]. □

Remark 2.1.5. In fact, any set X with cardinality n such that $2^n = |B|$ will do.

In essence, we can thus simply regard computations in finite Boolean Algebras as set-theoretic computations in the powerset of any set X with the required cardinality. So, we may regard finite Boolean Algebras as familiar objects.

2.2 Models

We are now ready to meet a key concept in Mathematical Logic, known as "structures", which can be seen (at least in some sense) as a generalisation of different algebraic concepts, like Groups, Rings, Vector Spaces, etc. There is an entire field devoted to the study of structures; this is known as Model Theory. In general, Model Theory studies the relationships between logical sentences and the structures in which they hold. Today, this field, in terms of applications outside of Mathematical Logic, has been used to prove results in Algebraic Geometry.

In essence, a structure is a set with some relations and functions defined on it, as well as possible distinguished elements, known as constants. We identify the *kind* of structure by the symbols of the relations, functions, and constants; and given this set of symbols, known as a vocabulary, we call a structure a τ-structure, where τ is the structure's vocabulary; and we denote the vocabulary of a structure $\mathfrak{M}$ by $\mathrm{Vocab}(\mathfrak{M})$.

We shall now give the official definitions of "vocabulary" and "structure", which I have pastiched together from [Hod97] and [Mar02]. Then, we shall see some examples.

Definition 2.2.1 (Vocabulary). A *vocabulary* is a (possibly empty)[1] set τ of function symbols, relation symbols, and constant symbols. Assigned to each function symbol and to each relation symbol

[1] In the context of First-Order Logic; Boolean Logic bans empty vocabularies.

is a positive integer known as its *arity*; a function symbol with arity n is a symbol corresponding to an n-place function – this holds, likewise, with relations.

Remark 2.2.2. For the purposes of this project, unless otherwise noted, we assume that all vocabularies are, at most, countable sets; and we assume that the relations and functions have a finite arity, but this can be relaxed if required.

Now, we have seen the definition of a "vocabulary", we can examine some examples:

- $\{\cdot, 1\}$ – this is the *vocabulary of Groups*; '$\cdot$' is our binary function symbol (used to represent Group multiplication), and '1' is our constant symbol (used to represent the Group identity);
- $\{\cdot, 1, {}^{-1}\}$ – this is an alternative vocabulary of Groups; in addition to our binary function symbol, and our constant symbol, we add a unary function symbol '${}^{-1}$' (used to represent Group inverses);
- $\{\cdot, +, 1, 0\}$ – this is the *vocabulary of Rings*; we have our binary relation symbols '$\cdot$' and '+', Ring multiplication and addition, respectively, and we have '1' and '0', the multiplicative and additive identities;
- $\{\wedge, \vee, \neg, \top, \bot\}$ – this is the *vocabulary of Boolean Algebras*; we have our two 2-place function symbols '$\wedge$' and '$\vee$' (representing meet and join, respectively), we have our 1-place function symbol '$\neg$' (representing complement), and we have our two constant symbols '$\top$' and '$\bot$' (representing top and bottom, respectively); and
- $\{<\}$ – this is the *vocabulary of orders*; we have only a single binary relation symbol '$<$'.

It is useful to be able to take from a vocabulary the sets of relations, functions, and constants individually; and, in fact, we can do one better, as seen in our next definition:

Definition 2.2.3. If τ is a vocabulary, we write $\mathrm{Const}(\tau)$ for the subset of τ containing only and all the constant symbols in τ. Similarly, for each $n \in \omega$, we write $\mathrm{Rel}_n(\tau)$ for the subset of τ containing only and all the n-ary relation symbols in τ. Again, for each $n \in \omega$, we write $\mathrm{Func}_n(\tau)$ for the subset of τ containing only and all the n-ary function symbols in τ.

We are now ready to meet structures, the main object of study in Model Theory.

Definition 2.2.4 (Structure). Given a vocabulary τ, a τ*-structure* $\mathfrak{M}$ is a non-empty set M, denoted $\mathrm{Dom}(\mathfrak{M})$, called the *domain* of $\mathfrak{M}$ together with

- for each constant symbol $c \in \mathrm{Const}(\tau)$, a distinguished element c' of M;
- for each positive integer n, and each n-ary function symbol $f \in \mathrm{Func}_n(\tau)$, an n-ary function $f' : M^n \to M$; and
- for each positive integer n, and each n-ary relation symbol $R \in \mathrm{Rel}_n(\tau)$, an n-ary relation on M.

Given a structure, we can define a function which picks out the corresponding constant, function, or relation to each symbol of the vocabulary:

Definition 2.2.5 (Interpretation). Let τ be a vocabulary, and $\mathfrak{M}$ a τ-structure. We define an *interpretation function* $\iota_{\mathfrak{M}} : \tau \to \mathfrak{M} \setminus \{\mathrm{Dom}(\mathfrak{M})\}$, which takes in a symbol s of the vocabulary τ, and gives us the 'interpretation' of that symbol s in $\mathfrak{M}$.

We shall now see some examples of structures.

- Let $\tau = \{\cdot, 1\}$, the vocabulary of Groups. Then, the Group K_4 defined by the following Cayley table is a τ-structure:

$\iota_{K_4}(\cdot)$	$\iota_{K_4}(1)$	a	b	c
$\iota_{K_4}(1)$	$\iota_{K_4}(1)$	a	b	c
a	a	$\iota_{K_4}(1)$	c	b
b	b	c	$\iota_{K_4}(1)$	a
c	c	b	a	$\iota_{K_4}(1)$

- Again, let $\tau = \{\cdot, 1\}$, the vocabulary of Groups. Then, if $\mathfrak{M}$ is a structure, with domain $M = \{x, y, z\}$, $\iota_{\mathfrak{M}}(1) = x$, and $\iota_{\mathfrak{M}}(\cdot)$ being the function that maps any pair from M^2 to $x \in M$, with no other constants relations, or functions, $\mathfrak{M}$ is a τ-structure. Note that this is not a Group.

- Let $\tau = \{\cdot, +, 1, 0\}$, the vocabulary of Rings. Then, $\mathbb{Z}$ with the usual addition and multiplication, and with distinguished elements $0, 1 \in \mathbb{Z}$, is a τ-structure.

- Let $\tau = \{\wedge, \vee, \neg, \top, \bot\}$, the vocabulary of Boolean Algebras. Then, if X is a finite set, $\mathcal{P}(X)$ together with union, intersection, complement in $\mathcal{P}(X)$, X, and $\emptyset$ is a τ-structure – and, following Theorem 2.1.4, a Boolean Algebra.

- Let $\tau = \{<\}$, the vocabulary of orders. Then any non-empty set M together with a binary relation R on M, defined to hold for any pair of elements of M is a τ-structure if we interpret the symbol '$<$' as R, and we have no further constants, functions, or relations.

It is oft convenient to quickly write a structure, given pre-existing definitions of constants, relations, and functions. This motivates the following notation: given a set M, and if $c_1, c_2, ..., c_m$ are constants, $f_1, f_2, ..., f_n$ are operators (functions from (tuples of) and to M) on M, and $R_1, R_2, ..., R_k$ are relations on M, we can write the given structure as $\langle M; c_1, c_2, \ldots, c_m, f_1, f_2, \ldots, f_n, R_1, R_2, \ldots, R_k\rangle$. For example, we can write the structure corresponding to the Ring $\mathbb{Z}$ as $\langle \mathbb{Z}; \cdot, +, 0, 1\rangle$, where $\cdot$ and $+$ are defined in their usual way.

Sometimes, it is necessary to restrict ourselves to just a portion of a structure; and to do that, we use the following definition:

Definition 2.2.6 (Reduct). Let τ be a vocabulary, and $\mathfrak{M}$ a τ-structure. If $\sigma \subseteq \tau$ is a vocabulary, and $\mathfrak{M}$ is a τ-structure, then we define $\mathfrak{M} \restriction \sigma$, the σ-*reduct* of $\mathfrak{M}$ as the σ-structure that is obtained by removing all of the constants, functions, and relations in $\mathfrak{M}$, which come from the vocabulary τ, but not the vocabulary σ.

For example, if $\tau = \{\cdot, 1\}$, the vocabulary of Groups, and $\sigma = \{\cdot\}$, then as $\sigma \subseteq \tau$, if $\mathfrak{M} := \langle G; +, e\rangle$ is a τ-structure, the σ reduct of $\mathfrak{M}$, $\mathfrak{M} \restriction \sigma = \langle G; +\rangle$.

Just as we are interested in removing constants, functions, and relations corresponding to fragments of our vocabulary, we are sometimes interested in expanding structures, by adding new constants, functions, and relations corresponding to symbols not appearing in our original vocabulary. This motivates the following definition.

Definition 2.2.7. Let σ and τ be vocabularies such that $\sigma \cap \tau = \emptyset$, and $\mathfrak{M}$ and $\mathfrak{N}$ be σ and τ-structures, respectively, with $\mathrm{Dom}(\mathfrak{M}) = \mathrm{Dom}(\mathfrak{N})$. Then we write $\mathfrak{M} \sqcup \mathfrak{N}$ to denote the $(\sigma \cup \tau)$- structure obtained by adding all of the constants, functions, and relations from $\mathfrak{N}$ into $\mathfrak{M}$ (or, indeed, *vice versa*).

For example, consider the $\{\cdot, 1\}$-structure $\langle \mathbb{Z}; +, 0\rangle$ (the additive Group), and the $\{<\}$-structure $\langle \mathbb{Z}; <\rangle$ ($\mathbb{Z}$ with its usual ordering), then we can combine the two structures (using the $\sqcup$ operation) to obtain a $\{\cdot, <, 1\}$-structure $\langle \mathbb{Z}; +, <, 0\rangle$ – the additive ordered Group $\mathbb{Z}$.

There is one more large topic to be introduced in this subsection – isomorphisms. Just as we have Group isomorphisms, order isomorphisms, Ring isomorphisms, &c., we can define τ-isomorphisms, where τ is any vocabulary.

Definition 2.2.8 (τ-isomorphism). Let τ be a vocabulary, and $\mathfrak{M}$ and $\mathfrak{N}$ be τ-structures. Then we call a bijection $\phi : \mathrm{Dom}(\mathfrak{M}) \to \mathrm{Dom}(\mathfrak{N})$ a *τ-isomorphism* if the following conditions hold:

- for each $c \in \mathrm{Const}(\tau)$, $\phi(\iota_{\mathfrak{M}}(c)) = \iota_{\mathfrak{N}}(c)$;
- for each positive integer n, each $f \in \mathrm{Func}_n(\tau)$, and each n-tuple $(m_1, m_2, \dots, m_n) \in \mathrm{Dom}(\mathfrak{M})$, $\phi(\iota_{\mathfrak{M}}(f)(m_1, m_2, \dots, m_n)) = \iota_{\mathfrak{N}}(f)(\phi(m_1), \phi(m_2), \dots, \phi(m_n))$; and
- for each positive integer n, each $R \in \mathrm{Rel}_n(\tau)$, and each n-tuple $(m_1, m_2, \dots, m_n) \in \mathrm{Dom}(\mathfrak{M})$, $\iota_{\mathfrak{M}}(R) m_1 m_2 \cdots m_n$ if and only if $\iota_{\mathfrak{N}}(R)\phi(m_1)\phi(m_2)\cdots\phi(m_n)$.

We adopt the standard notation to show the existence of an isomorphism between two τ-structures:

Definition 2.2.9. If two τ-structures $\mathfrak{M}$ and $\mathfrak{N}$ are isomorphic, we write $\mathfrak{M} \cong \mathfrak{N}$.

For example, if we have two $\{\cdot, 1\}$-structures, which are Groups, and they are Group isomorphic, by ϕ, then ϕ is also a $\{\cdot, 1\}$-isomorphism.

However, we can also have $\{\cdot, 1\}$-isomorphisms, which are not Group isomorphisms, if neither of the $\{\cdot, 1\}$-structures are Groups. This happens in the following case: we have $\{\cdot, 1\}$-structure $\mathfrak{M} := \langle M := \{x, y, z\}; f : M \times M \to M[(a, b) \mapsto x], x\rangle$, then the bijection $\phi : M \to M$ defined by $x \mapsto x$, $y \mapsto z$, and $z \mapsto y$ is a $\{\cdot, 1\}$-isomorphism, because $\phi(\iota_{\mathfrak{M}}(1)) = \phi(x) = x = \iota_{\mathfrak{M}}(1)$, and, for arbitrary a and b in M, $\phi(\iota_{\mathfrak{M}}(\cdot)(a, b)) = \phi(f(a, b)) = \phi(x) = x = f(\phi(a), \phi(b)) = \iota_{\mathfrak{M}}(\cdot)(\phi(a), \phi(b))$.

Finally, we introduce a final definition, before we proceed onto the next section on Logic:

Definition 2.2.10. Let σ and τ be vocabularies. If $\cdot^* : \tau \to \sigma$ is a bijection, and maps constant symbols to constant symbols, n-ary function symbols to n-ary function symbols, and n-ary relation symbols to n-ary relation symbols, then we call $\cdot^*$ a *renaming*, and given a τ-structure $\mathfrak{M}$, we can write $\mathfrak{M}^*$ to represent the corresponding σ-structure, which arises as the result of replacing all the symbols of τ by the corresponding symbols (under $\cdot^*$) in σ.

For example, if $\tau = \{\cdot, 0\}$ and $\sigma = \{+, 1\}$. Then, the function $\cdot^* : \tau \to \sigma$ defined by $\cdot \mapsto +$ and $0 \mapsto 1$ is a renaming, and if $\mathfrak{M}$ is a τ-structure, the renamed structure $\mathfrak{M}^* = \langle \mathrm{Dom}(M); \iota_{\mathfrak{M}}(\cdot), \iota_{\mathfrak{M}}(0)\rangle$, and $\iota_{\mathfrak{M}^*}(+) = \iota_{\mathfrak{M}}(\cdot)$ and $\iota_{\mathfrak{M}^*}(0) = \iota_{\mathfrak{M}}(1)$.

2.3 Logic

We shall now meet the notion of "Logic" (I take the specific definition from [Ebb16]). We will then prove that we can construct, from Boolean Algebras, a class of "Logics".

Intuitively, a Logic, is a method of testing for "truth in a structure"; we are given a list of properties that we can check for, and a method of checking if they hold. The following definition seeks to generalise this notion.

Definition 2.3.1 (Logic). A *Logic* is a function $\mathcal{L}$ from vocabularies to strings of symbols (known as $\mathcal{L}$-sentences), together with a relation $\vDash_{\mathcal{L}}$ between structures and $\mathcal{L}$-sentences (we read '$\vDash_{\mathcal{L}}$' as 'models in $\mathcal{L}$' – and, we often write $\mathcal{L}$ to mean the entire function-relation pair) such that the following properties hold:

(i) if $\tau \subseteq \sigma$, then $\mathcal{L}(\tau) \subseteq \mathcal{L}(\sigma)$;

(ii) if $\mathfrak{M} \vDash_{\mathcal{L}} \phi$, then $\phi \in \mathcal{L}(\mathrm{Vocab}(\mathfrak{M}))$;

(iii) (*the isomorphism property*) if $\mathfrak{M} \vDash_{\mathcal{L}} \phi$ and $\mathfrak{M} \cong \mathfrak{N}$, then $\mathfrak{N} \vDash_{\mathcal{L}} \phi$;

(iv) (*the reduct property*) if $\phi \in \mathcal{L}(\tau)$ and $\tau \subseteq \mathrm{Vocab}(\mathfrak{M})$, then $\mathfrak{M} \vDash_{\mathcal{L}} \phi$ if and only if $\mathfrak{M} \restriction \tau \vDash_{\mathcal{L}} \phi$; and

(v) (*the renaming property*) if $\cdot^* : \tau \to \sigma$ is a renaming, then for every $\phi \in \mathcal{L}(\tau)$, there is a sentence $\phi' \in \mathcal{L}(\sigma)$ such that for any τ-structure $\mathfrak{M}$, $\mathfrak{M} \vDash_{\mathcal{L}} \phi$ if and only if $\mathfrak{M}^* \vDash_{\mathcal{L}} \phi'$

Informally, we take "ϕ being modelled by $\mathfrak{M}$ in $\mathcal{L}$" as being some truth-claim about ϕ in $\mathfrak{M}$, with respect to the Logic $\mathcal{L}$; and, again, informally, each of the above conditions has the following meaning, respectively:

(i) every $\mathcal{L}$-sentence is determined by some (possibly none) symbols in a given vocabulary;

(ii) a given structure $\mathfrak{M}$ can only model (in $\mathcal{L}$) $\mathcal{L}$-sentences that depend on the vocabulary of $\mathfrak{M}$;

(iii) the modelling relation is invariant under isomorphism;

(iv) if a $\mathcal{L}$-sentence ϕ is true in a structure $\mathfrak{M}$ with respect to the Logic $\mathcal{L}$, then it is true only in virtue of the relationships between the interpretations of the symbols upon which ϕ is determined by; and

(v) after renaming a structure, there is a new renamed $\mathcal{L}$-sentence with the same meaning as any $\mathcal{L}$-sentence before the renaming.

So, this definition of "Logic" is trying to capture the key notions of what it means for a concept of "truth in a structure". We shall now see our first example of a Logic: a Boolean Algebra (or, as we shall call it from now on, a "Boolean Logic"). We will begin by defining our Boolean-Logic-Sentences, and once we have seen it, we will define how to check for "truth in a structure" given a Boolean Logic. The definition is inductive, and later on, we shall use (almost – there shall be only two exceptions) the same definition for the sentences of First-Order Logic.

Sometimes we read $\mathfrak{M} \vDash_{\mathcal{L}} \phi$ as "$\mathfrak{M}$ satisfies ϕ" or as "ϕ is true in $\mathfrak{M}$" (according to the Logic $\mathcal{L}$). We also use, as shorthand, where X is a set of $\mathcal{L}(\tau)$-sentences, $\mathfrak{M} \vDash_{\mathcal{L}} X$ to mean for each $\phi \in X$, $\mathfrak{M} \vDash_{\mathcal{L}} \phi$.

2.3.1 Sentences of Boolean Logic

We have a (never-ending) supply of *variables*. Typically, these are denoted by x, y, x_1, x_2, y_1, y_2, &c. Our simplest "subsentential-unit" is a *term*, these show up in sentences, but are never sentences themselves.

Definition 2.3.2 (Terms). Given a vocabulary τ, we define the *terms of Boolean Logic* as follows:

- every variable;
- every constant symbol in $\mathrm{Const}(\tau)$;
- for each $n \in \mathbb{Z}^+$, if $f \in \mathrm{Func}_n(\tau)$, and if $t_1, \dots, t_n$ are terms, then so is $f(t_1, \dots, t_n)$; and
- that's all.

Let $\tau = \{\cdot, 1\}$ be the vocabulary of Groups. Then, we have only one term (of the second kind) of Boolean Logic, given τ, namely 1, because $1 \in \mathrm{Const}(\tau)$, and is the only constant symbol in τ. We know, that by the first bullet point, that each variable is a term, and by the third, we also know that $1 \cdot 1$ is a term, and so are $x \cdot 1$ (where x is a variable) and $(1 \cdot 1) \cdot 1$. More precisely, we should write $\cdot(x, y)$ instead of $x \cdot y$, by the above definition, but it is easy to see what is meant, so we stick to the normal convention.

The idea of a term (given a vocabulary τ), is that it somehow refers to – or, "picks out" – some element of the τ-structures.

Using these terms, we can build up to our simplest sentential unit, known as an "atomic sentence", but first, we need to define a slight generalisation of "sentences", known as "formulæ", and we begin with "atomic formulæ", which are the simplest formulæ.

Definition 2.3.3 (Atomic Formulæ). Given a vocabulary τ, we define the *atomic formulæ of Boolean Logic* as follows:

- if t_1 and t_2 are terms of Boolean Logic, given τ, then $t_1 = t_2$ is an atomic formula (of Boolean Logic, given τ);
- for each $n \in \mathbb{Z}^+$, if $R \in \mathrm{Rel}_n(\tau)$, and if $t_1, \dots, t_n$ are terms of Boolean Logic, given τ, then $Rt_1 \cdots t_n$ is an atomic formula (of Boolean Logic, given τ); and
- that's all.

Following our previous example, where $\tau = \{\cdot, 1\}$, we can look at some of the atomic formulæ of Boolean Logic, given τ: given the first bullet point, we know that $1 = 1$, $x = 1$, $1 = x$, and $x = y$ are all atomic formulæ of Boolean Logic, given τ, where x and y are variables, because 1, x, and y are terms. Given τ, we have no atomic formulæ of the kind specified by the second bullet point, because there are no relation symbols in τ. If, however, we expand τ, by defining a new vocabulary to also include the binary relation symbol $<$, so $\tau' := \tau \cup \{<\}$ (which is the vocabulary of ordered Groups), then we will have some atomic formulæ of the second kind (given τ'). In particular, here are some examples of atomic formulæ of Boolean Logic (given τ') are (where x and y are variables): $1 < x$, $(1 \cdot 1) < 1$, and $(1 \cdot x) < (x \cdot (x \cdot y))$. Again, notice how we write $x < y$, rather than $< xy$, as our definition strictly calls for.

I said that atomic formulæ were a generalisation of atomic sentences; so, we shall now see the definition of an atomic sentence. We say that an atomic formula of Boolean Logic (given a vocabulary τ) ϕ is an atomic sentence of Boolean Logic (given a vocabulary τ) if and only if ϕ contains no variables. The idea behind atomic sentences is that there is a direct way to check whether they are "true in a structure", whereas to check whether a non-atomic sentence is "true in a structure", we rely on the facts about the truth of the atomic sentences. That is to say that the truth of non-atomic sentences supervenes on the truth of the atomic sentences in a given structure. This idea shall be made explicit when we define our "models" relation for Boolean Logics (and later for First-Order and Infinitary Logics).

We often write "terms", "sentences", "formulæ", *&c.*, when the vocabulary and Logic in question is clear.

We shall now define our direct way of checking whether an atomic sentence holds (is satisfied) in a structure.

Definition 2.3.4. Let $\mathfrak{M}$ be a τ-structure, then there exists a function $\nu_{\mathfrak{M}}$ which takes an atomic sentence of Boolean Logic, given the vocabulary τ, and outputs either 0 or 1 depending on whether the atomic sentence holds in $\mathfrak{M}$ or not.

Before explicitly defining this function, however, we first give a function which, when given a term returns the element of the domain which the term should be interpreted to, and when given a relation, gives the corresponding relation of the structure. We shall call this function $\nu^\star_{\mathfrak{M}}$ for a given τ-structure $\mathfrak{M}$. We will inductively define this function, given a term, without variables, of Boolean Logic (given the vocabulary τ) or given some $R \in \bigcup_{i\in\mathbb{Z}^+} \mathrm{Rel}_i(\tau)$:

- if t is a term of the form c for $c \in \mathrm{Const}(\tau)$, then $\nu^\star_{\mathfrak{M}}(t) = \iota_{\mathfrak{M}}(c)$;
- if t is a term of the form $f(t_1, \dots, t_n)$, where $t_1, \dots, t_n$ are terms, and $f \in \mathrm{Func}_n(\tau)$, then $\nu^\star_{\mathfrak{M}}(t) = \iota_{\mathfrak{M}}(f)(\nu^\star_{\mathfrak{M}}(t_1), \dots, \nu^\star_{\mathfrak{M}}(t_n))$; and
- if $R \in \bigcup_{i\in\mathbb{Z}^+} \mathrm{Rel}_i(\tau)$, then $\nu^\star_{\mathfrak{M}} = \iota_{\mathfrak{M}}(R)$.

We can now inductively define $\nu_{\mathfrak{M}}$ as follows, given a τ-structure $\mathfrak{M}$ and an atomic sentence of Boolean Logic (given τ) ϕ:

- If ϕ is of the form $t_1 = t_2$, where t_1 and t_2 are terms, then $\nu_{\mathfrak{M}}(\phi) = 1$ if and only if (and 0 otherwise) $\nu^\star_{\mathfrak{M}}(t_1) = \nu^\star_{\mathfrak{M}}(t_2)$; and
- If ϕ is of the form $Rt_1 \cdots t_n$ for terms $t_1, \dots, t_n$, then $\nu_{\mathfrak{M}}(\phi) = 1$ if and only if (and 0 otherwise) the relation $\nu^\star_{\mathfrak{M}}(R)\nu^\star_{\mathfrak{M}}(t_1) \cdots \nu^\star_{\mathfrak{M}}(t_n)$ holds.

For example, we will let $\tau = \{\cdot, <, 1\}$ be the vocabulary of Ordered Groups, and we will consider the τ-structure $\langle \mathbb{Z}; +, <, 0\rangle$ (the additive Group with the usual ordering on the integers). Then, clearly $\phi :=< \cdot(1,1) \cdot (1, \cdot(1,1))$ is an atomic sentence of Boolean Logic, given τ, and so we can calculate $\nu_{\mathfrak{M}}(\phi)$:

$$\begin{aligned}
\nu_{\mathfrak{M}}(\phi) = 1 &\Leftrightarrow \nu^\star_{\mathfrak{M}}(<)\nu^\star_{\mathfrak{M}}(\cdot(1,1))\nu^\star_{\mathfrak{M}}(\cdot(1,\cdot(1,1))) \text{ holds}\\
&\Leftrightarrow (\nu^\star_{\mathfrak{M}}(1) + \nu^\star_{\mathfrak{M}}(1)) < (\nu^\star_{\mathfrak{M}}(1) + (\nu^\star_{\mathfrak{M}}(\cdot(1,1))) \text{ holds}\\
&\Leftrightarrow (0+0) < (0 + (\nu^\star_{\mathfrak{M}}(1) + \nu^\star_{\mathfrak{M}}(1)) \text{ holds}\\
&\Leftrightarrow 0 < (0 + (0+0)) \text{ holds}\\
&\Leftrightarrow 0 < 0
\end{aligned}$$

And, so, $\nu_{\mathfrak{M}}(\phi) = 0$.

We shall now see how, using atomic formulæ, we can build up to "full formulæ", and from there we can define, analogously to atomic sentences, "full sentences".

Definition 2.3.5 (Formulæ). Given a vocabulary τ, we define the *formulæ of Boolean Logic* as follows:

- if ϕ is an atomic formula of Boolean Logic, given τ, then ϕ is a formula;
- if ϕ and ψ are formulæ of Boolean Logic, given τ, then $(\phi \wedge \psi)$ is also;
- if ϕ and ψ are formulæ of Boolean Logic, given τ, then $(\phi \vee \psi)$ is also;
- if ϕ is a formula of Boolean Logic, given τ, then $\neg\phi$ is also; and
- that's all.

We often omit brackets inside formulæ, where the meaning is still clear; and we almost always omit any pair of brackets that would be the outermost symbols in the formula (sometimes we add brackets too, where the meaning would be more clear).

Given the vocabulary of Groups, $\tau = \{\cdot, 1\}$, we know that the following are formulæ of Boolean Logic (where x and y are variables): $((x \cdot y) < 1) \wedge x = y$, $x < 1$, and $(1 < 1) \vee \neg(x < 1 \wedge 1 < (x \cdot y))$.

Again, we define the *sentences of Boolean Logic* (given a vocabulary τ) to be the formulæ of Boolean Logic (given τ) such that no variables appear in them. We can see a hint of the idea that atomic sentences depend only on the structure itself, whereas non-atomic sentences are built up from these units, given this definition, as the atomic sentences are our only "basis case" in our inductive definition. We can also see that our definition of the sentences of Boolean Logic mirrors that of a Boolean Algebra: if we were somehow able to map all the atomic sentences into elements of a Boolean Algebra, then every formula would be an expression in the given Boolean Algebra, and so would be equal to some element of the Boolean Algebra. This is the idea that we will use, shortly, to define Boolean Logic.

Given the language of ordered Groups, $\tau = \{\cdot, 1, <\}$, all of these sentences of Boolean Logic, given τ are the atomic formulæ, where no variables appear, but only the constant 1. For example, $((1 \cdot 1) < 1) \wedge 1 = 1$, $1 < 1$, and $(1 < 1) \vee \neg(1 < 1 \wedge 1 < (1 \cdot 1))$.

The reason that formulæ are a generalisation of sentences, is that we can view formulæ as functions, from the set of constants of the relevant vocabulary onto sentences of the relevant vocabulary. We simply have to define which variables map to which constants. We can see that, by setting $x, y = 1$, we obtain the sentences $((1 \cdot 1) < 1) \wedge 1 = 1$, $1 < 1$, and $(1 < 1) \vee \neg(1 < 1 \wedge 1 < (1 \cdot 1))$ from the formulæ $((x \cdot y) < 1) \wedge x = y$, $x < 1$, and $(1 < 1) \vee \neg(x < 1 \wedge 1 < (x \cdot y))$. There is a standard notation for this. If ϕ is a formula whose variables are among a set of variables $V = \{v_1, v_2, \ldots, v_n\}$, then we can write $\phi(v_1, v_2, \ldots, v_n)$ to signify this fact, and if $c_1, c_2, \ldots, c_n$ is a sequence of constants of the relevant language, then we can write $\phi(c_1, c_2, \ldots, c_n)$ to obtain the sentence resulting from replacing all the occurrences of the variable v_1 by the constant symbol c_1, and all the occurrences of the variable v_2 by the constant symbol c_2, and so on. For example, if ϕ is $((x \cdot y) < 1) \wedge x = y$, then we can write $\phi(x, y)$ to signify that ϕ's variable are among $\{x, y\}$, and then $\phi(1, 1)$ is $((1 \cdot 1) < 1) \wedge 1 = 1$.

2.3.2 Boolean Logic Modelling Relation

We have now seen enough information to be able to define our 'models' relation for Boolean Logic. We will, as said before, map all of the atomic sentences onto elements of the Boolean Algebra, and, given a sentence, we can, after the mapping of atomic sentences, evaluate the resulting expression of the Boolean Algebra, and then, we shall check if this value is in a special set, called the designated values; we view this set as the set of elements of the Boolean Algebra that tell us that a sentence is a true. We shall see this, formally, in our next definition.

Definition 2.3.6 (Boolean Logic). Given a Boolean Algebra B, a subset $D \subseteq B$ (the *designated values*), and a mapping μ which takes any atomic sentence of Boolean Algebra (given any vocabulary), and maps it so some element of B, we can define the *Boolean Logic* $\mathcal{B}_{D,\mu}$, which consists of a function $\mathbb{B}$, defined on vocabularies, that gives us all the Sentences of Boolean Logic, given the vocabulary τ; and a relation $\vDash_{\mathcal{B}_{D,\mu}}$ between structures and $\mathcal{B}_{D,\mu}$-sentences, and is defined, using another function μ' (which we shall define after), as follows (given a vocabulary τ, a τ-structure $\mathfrak{M}$, and some $\phi \in \mathbb{B}(\tau)$):

$$\mathfrak{M} \vDash_{\mathcal{B}_{D,\mu}} \phi \quad \text{if and only if} \quad \mu'(\phi) \in D.$$

We will now define μ', inductively: which maps sentences into elements of the Boolean Algebra (given a vocabulary τ, a τ-structure $\mathfrak{M}$, and some $\phi \in \mathbb{B}(\tau)$):

- if ϕ is an atomic sentence, then $\mu'(\phi) = \mu(\phi)$ if $\nu_{\mathfrak{M}}(\phi) = 1$, and $\mu'(\phi) = \neg\mu(\phi)$ otherwise (the '$\neg$' is Boolean complement);
- if ϕ is of the form $\psi \wedge \chi$, then $\mu'(\phi) = \mu'(\psi) \wedge \mu'(\chi)$ (that is, our Boolean meet on the right of the equality);
- if ϕ is of the form $\psi \vee \chi$, then $\mu'(\phi) = \mu'(\psi) \vee \mu'(\chi)$ (that is, our Boolean join on the right of the equality); and
- if ϕ is of the form $\neg\psi$, then $\mu'(\phi) = \neg\mu'(\psi)$ (that is, our Boolean complement on the right of the equality).

For an example, we will use the following Boolean Algebra B: the domain is $\{\top, \bot, \vdash, \dashv\}$ (top, bottom, left and right), and our operations are defined as:

$\wedge$	$\top$	$\vdash$	$\dashv$	$\bot$
$\top$	$\top$	$\vdash$	$\dashv$	$\bot$
$\vdash$	$\vdash$	$\vdash$	$\bot$	$\bot$
$\dashv$	$\dashv$	$\bot$	$\dashv$	$\bot$
$\bot$	$\bot$	$\bot$	$\bot$	$\bot$

$\vee$	$\top$	$\vdash$	$\dashv$	$\bot$
$\top$	$\top$	$\top$	$\top$	$\top$
$\vdash$	$\top$	$\vdash$	$\top$	$\vdash$
$\dashv$	$\top$	$\top$	$\dashv$	$\dashv$
$\bot$	$\top$	$\vdash$	$\dashv$	$\bot$

	$\top$	$\vdash$	$\dashv$	$\bot$
$\neg$	$\bot$	$\dashv$	$\vdash$	$\bot$

Our designated values D, will be $\top$ and $\vdash$. We will consider the vocabulary $\tau = \{\cdot, <, 1\}$, of Ordered Groups. Our τ-structure $\mathfrak{M}$ will be $\langle \mathbb{Z}; +, <, 0 \rangle$ (that is, the additive Group, with the usual ordering on the integers). We will only define some values of μ that are relevant to us. We shall set $\mu(1 = 1) = \top$ and $\mu(1 < 1) = \dashv$.

Then, we consider the sentence $\phi := (1 = 1) \wedge \neg(1 < 1)$.

$$
\begin{aligned}
&\mathfrak{M} \vDash_{\mathcal{B}_{D,\mu}} \phi\\
\Leftrightarrow\ & \mu'(\phi) \in D\\
\Leftrightarrow\ & (\mu'(1=1) \wedge \mu'(\neg(1<1))) \in D\\
\Leftrightarrow\ & (\mu(1=1) \wedge \neg\mu'(1<1)) \in D\\
\Leftrightarrow\ & (\top \wedge \neg\neg \dashv) \in D\\
\Leftrightarrow\ & (\top \wedge \neg \vdash) \in D\\
\Leftrightarrow\ & (\top \wedge \dashv) \in D\\
\Leftrightarrow & \dashv \in D,
\end{aligned}
$$

as $\nu_{\mathfrak{M}}(1=1) = 1$, and $\nu_{\mathfrak{M}}(1<1) = 0$; hence; $\mathfrak{M} \nvDash_{\mathcal{B}_{D,\mu}} \phi$.

We shall now provide a sketch of the proof[2] that Boolean Logics are indeed Logics.

Theorem 2.3.7. *Every Boolean Logic is a Logic.*

Sketch of proof: Let $\mathcal{B}_{D,\mu}$ denote a Boolean Logic on the Boolean Algebra B.

- Certainly, $\tau \subseteq \sigma$ implies that $\mathbb{B}(\tau) \subseteq \mathbb{B}(\sigma)$, from our definition;
- if $\mathfrak{M} \vDash_{\mathcal{B}_{D,\mu}} \phi$, then $\phi \in \mathbb{B}(\mathrm{Vocab}(\mathfrak{M}))$, again, by definition;
- it can be shown that if $\mathfrak{M} \cong \mathfrak{N}$, then $\nu_{\mathfrak{M}}(\phi) = \nu_{\mathfrak{N}}(\phi)$ for any atomic $\phi \in \mathbb{B}(\mathrm{Vocab}(\mathfrak{M}))$, and then, we can inductively prove the isomorphism property;
- the reduct property follows by our inductive definition: if $\mathfrak{M} \vDash_{\mathcal{B}_{D,\mu}} \phi$, and ϕ is a $\mathbb{B}(\tau)$-sentence then it can be shown that ϕ depends only and fully on $\mathbb{B}(\tau)$-atomic sentences, which, it can be shown that, (they) are described by the τ-reduct of $\mathfrak{M}$, and so the reduct property follows;
- the renaming property also immediately follows from the inductive definitions: the required sentence is just the result of applying the renaming to the original sentence.

■

If the reader is interested, they can prove these properties formally, given the above sketch.

We shall now define 'Logical strength':

Definition 2.3.8 (Logical Strength). Let $\mathcal{L}$ and $\mathcal{L}'$ be Logics. We say that $\mathcal{L}'$ is *as strong as* $\mathcal{L}$ (or, equivalently, call $\mathcal{L}$ a *Sublogic* of $\mathcal{L}'$), and write $\mathcal{L} \leq \mathcal{L}'$ if and only if for any vocabulary τ we have

$$\{\{\mathfrak{M};\ \mathfrak{M} \vDash_{\mathcal{L}} \phi\};\ \phi \in \mathcal{L}(\tau)\} \subseteq \{\{\mathfrak{M};\ \mathfrak{M} \vDash_{\mathcal{L}'} \phi\};\ \phi \in \mathcal{L}'(\tau)\};$$

Essentially, what the above definition says, is that we call $\mathcal{L}$ a Sublogic of $\mathcal{L}'$ if and only if for any formula $\mathcal{L}$-sentence ϕ, there is an $\mathcal{L}'$-sentence ϕ' such that ϕ and ϕ' are modelled by the same set of τ-structures. So, very informally, we say a Logic is a Sublogic of another when the 'stronger' Logic has

[2] All this background information, for the first "proof" to only be a sketch... Disappointing, I know.

at least as much expressive power, in terms of "picking out properties of structures", as the 'weaker' Logic.

If $\mathcal{L} \leq \mathcal{L}'$, and $\phi \in \mathcal{L}(\tau)$ for a given vocabulary τ, then we often write ϕ (yes; this is not a typo!) to denote an arbitrary $\mathcal{L}'$-sentence $\psi \in \mathcal{L}'(\tau)$, which is such that

$$\{\mathfrak{M};\ \mathfrak{M} \vDash_{\mathcal{L}} \phi\} = \{\mathfrak{M};\ \mathfrak{M} \vDash_{\mathcal{L}'} \psi\};$$

we do this because it is convenient to view the weaker Logic as genuinely embedded in the stronger.

Let C be an arbitrary Boolean Algebra, μ_1 be arbitrary, and $D_1 = \emptyset$, then resulting Boolean Logic $\mathcal{C}_{D_1,\mu_1}$ can be seen to, for any τ-structure, and any $\phi \in \mathbb{B}(\tau)$ to have $\mathfrak{M} \nvDash_{\mathcal{C}_{D_1,\mu_1}} \phi$, for there is no way for $\mu(\phi)$ to lie in D_1, as D_1 is empty. Thus, if we have another Logic $\mathcal{L}$, with a sentence that holds in no structures, then $\mathcal{C}_{D_1,\mu_1}$ is a Sublogic of $\mathcal{L}$ (we shall see that First-Order Logic is such a Logic, and so $\mathcal{C}_{D_1,\mu_1}$ is a Sublogic of First-Order Logic). A different Boolean Logic with such a sentence can be constructed as follows: let B be our four element Boolean Algebra defined on $\{\top, \bot, \vdash, \dashv\}$, and set $D_2 = \{\top\}$, and let μ_2 be such that for any vocabulary τ there exists atomic $\phi_\tau \in \mathbb{B}(\tau)$ such that $\mu_2(\phi_\tau) = \vdash$. Then, we see that for any τ-structure $\mathfrak{M}$, $\mathfrak{M} \nvDash_{\mathcal{B}_{D_2,\mu_2}} \phi_\tau$, because $\mu_2'(\phi_\tau) \in \{\vdash, \neg \vdash\} = \{\vdash, \dashv\}$, and $\{\vdash, \dashv\} \cap D_2 = \emptyset$, so no matter whether $\nu_{\mathfrak{M}}(\phi_\tau) = 0$ or $\nu_{\mathfrak{M}}(\phi_\tau) = 1$, $\mu_2'(\phi_\tau) = \mu_2(\phi_\tau) \notin D$.

Explicitly, then, for any vocabulary τ,

$$\{\{\mathfrak{M} \vDash_{\mathcal{C}_{D_1,\mu_1}} \phi\};\ \phi \in \mathbb{B}(\tau)\} = \{\emptyset;\ \phi \in \mathbb{B}(\tau)\} = \{\emptyset\} \subseteq \{\{\mathfrak{M} \vDash_{\mathcal{B}_{D_2,\mu_2}} \phi\};\ \phi \in \mathbb{B}(\tau)\},$$

as $\{\mathfrak{M} \vDash_{\mathcal{B}_{D_2,\mu_2}} \phi_\tau\} = \emptyset$.

And, so, $\mathcal{C}_{D_1,\mu_1} \leq \mathcal{B}_{D_2,\mu_2}$.

We also have a notion of when Logics are equivalent; that is, when they both have the same expressive strength (a notion that will be made explicit in Section 5). But, the definition is easy enough to understand, given one understands "Sublogic".

Definition 2.3.9. We say that two Logics $\mathcal{L}$ and $\mathcal{L}'$ are *equivalent* if and only if $\mathcal{L} \leq \mathcal{L}'$ and $\mathcal{L}' \leq \mathcal{L}$.

2.4 First-Order Logic

Just as we defined our sentences of Boolean Algebras (given a vocabulary τ), we can formally define the sentences of First-Order Logic (given a vocabulary τ). The resulting set of sentences will be familiar, as we are used to working with First-Order Logic. The sentences of First-Order Logic are almost the same as those of Boolean Algebras, but we also allow quantification (that is, we can say sentences such as "for all x, there is something bigger than x"), implication ("if x, then y"), and we define something to be false in all structures.

2.4.1 Sentences of First-Order Logic

We begin by defining our terms of First-Order Logic in the same way as before. Once again, we have a stock of variables: x, y, x_1, x_2, y_1, y_2, &c.

Definition 2.4.1 (Terms). Given a vocabulary τ, we define the *terms of First-Order Logic* as follows:

- every variable;
- every constant symbol in $\text{Const}(\tau)$;

- for each $n \in \mathbb{Z}^+$, if $f \in \mathrm{Func}_n(\tau)$, and if $t_1, \dots, t_n$ are terms, then so is $f(t_1, \dots, t_n)$; and
- that's all.

Our atomic formulæ (and consequently atomic sentences) are the same as before, but we also include "$\perp$", which we will take to always be false (that is, $\perp$ is not true in any structure).

Definition 2.4.2 (Atomic Formulæ). Given a vocabulary τ, we define the *atomic formulæ of First-Order Logic* as follows:

- $\perp$ is an atomic formula;
- if t_1 and t_2 are terms of First-Order Logic, given τ, then $t_1 = t_2$ is an atomic formula;
- for each $n \in \mathbb{Z}^+$, if $R \in \mathrm{Rel}_n(\tau)$, and if $t_1, \dots, t_n$ are terms of First-Order Logic, given τ, then $Rt_1 \cdots t_n$ is an atomic formula; and
- that's all.

Again, from our atomic formulæ we can build our formulæ, using a similar definition to that for Boolean Algebras, but we add some new cases: $\rightarrow$, for "implication", $\exists$ for "existential quantification", and $\forall$ for "universal quantification". So, in our structures, we can quantify over members of the domain, rather than always making reference to a specific member of the domain, via a constant, like in Boolean Algebras.

Definition 2.4.3 (Formulæ). Given a vocabulary τ, we define the *formulæ of First-Order Logic* as follows:

- if ϕ is an atomic formula of First-Order Logic, given τ, then ϕ is a formula;
- if ϕ and ψ are formulæ of First-Order Logic, given τ, then $(\phi \wedge \psi)$ is also;
- if ϕ and ψ are formulæ of First-Order Logic, given τ, then $(\phi \rightarrow \psi)$ is also;
- if ϕ and ψ are formulæ of First-Order Logic, given τ, then $(\phi \vee \psi)$ is also;
- if ϕ is a formula of First-Order Logic, given τ, then $\neg\phi$ is also;
- if ϕ is a formula of First-Order Logic, given τ, and v is a variable, then so is $\exists v\phi$;
- if ϕ is a formula of First-Order Logic, given τ, and v is a variable, then so is $\forall v\phi$; and
- that's all.

In practice, however, we treat $\phi \vee \psi$ as an abbreviation for $\neg(\neg\phi \wedge \neg\psi)$, $\phi \rightarrow \psi$ as an abbreviation for $\neg\phi \vee \psi$, and $\forall v\phi$ as an abbreviation for $\neg\exists v\neg\phi$. So, we can dispense of the clauses for $\vee$, $\rightarrow$, and $\forall$. Moreover, we treat $\phi \leftrightarrow \psi$ as an abbreviation for $(\phi \rightarrow \psi) \wedge (\psi \rightarrow \phi)$.

We also often omit (or add brackets) when the meaning is clear; we typically omit a pair of brackets if they are the outermost symbols.

Given a formula ϕ of First-Order Logic, we say a variable v is *bound* if all occurrences of v occur in subformulæ[3] of ϕ of the forms $\exists v\psi$ or $\forall v\psi$. If a formula in ϕ is not bound, we say it is *free*. Our

[3] A formula is a subformula of another if it appears, in totality, without any breaks in the latter.

sentences of First-Order Logic, given a vocabulary τ are precisely the set of formulæ of First-Order Logic that contain no free variables. This, is in contrast to Boolean Logics, where our sentences contained no variables whatsoever.

For example, in the formula $(x \vee y)$, x and y are free; but in the formula $\exists x \forall y(x \vee y)$, both x and y are bound. Therefore, the latter is a sentence, but the former is not.

We can, in some sense, treat formulas like functions: if ϕ is a formula, whose free variables are among $v_1, v_2, \dots, v_n$, then we can write $\phi(v_1, v_2, \dots, v_n)$ to signify this fact. Then, if $c_1, c_2, \dots, c_n$ are constant symbols, we can write $\phi(c_1, c_2, \dots, c_n)$ to obtain the sentence that results from replacing the free occurrences of the variable v_i with the constant symbol c_i. For example, if $\phi(x, y)$ is $(x \vee y)$, then $\phi(c_1, c_2)$ is $(c_1 \vee c_2)$.

2.4.2 Truth in First-Order Logic

Definition 2.4.4. We denote First-Order Logic as $\mathcal{L}_{\omega,\omega}$. We shall see why in the next section.

Definition 2.4.5 (First-Order Logic). First-Order Logic (denoted $\mathcal{L}_{\omega,\omega}$) is the function $\mathcal{L}_{\omega,\omega}$ that returns the set of Sentences of First-Order Logic, given a vocabulary τ together with a relation $\vDash_{\mathcal{L}_{\omega,\omega}}$ between structures and $\mathcal{L}_{\omega,\omega}$-sentences, which is defined, inductively, as follows (for a vocabulary τ, a $\mathcal{L}_{\omega,\omega}(\tau)$-sentence ϕ, and a τ-structure $\mathfrak{M}$):

- for $\phi = \bot$, then $\mathfrak{M} \nvDash_{\mathcal{L}_{\omega,\omega}} \phi$;
- for atomic (excluding $\bot$) ϕ, then $\mathfrak{M} \vDash_{\mathcal{L}_{\omega,\omega}} \phi$ if and only if $\nu_{\mathfrak{M}}(\phi) = 1$;
- for ϕ of the form $\neg\psi$, then $\mathfrak{M} \vDash_{\mathcal{L}_{\omega,\omega}} \phi$ if and only if $\mathfrak{M} \nvDash_{\mathcal{L}_{\omega,\omega}} \psi$;
- for ϕ of the form $\psi \wedge \chi$, then $\mathfrak{M} \vDash_{\mathcal{L}_{\omega,\omega}} \phi$ if and only if $\mathfrak{M} \vDash_{\mathcal{L}_{\omega,\omega}} \psi$ and $\mathfrak{M} \vDash_{\mathcal{L}_{\omega,\omega}} \chi$; and
- for ϕ of the form $\exists v \psi(v)$, then $\mathfrak{M} \vDash_{\mathcal{L}_{\omega,\omega}} \phi$ if and only if $\mathfrak{M} \sqcup \langle \mathrm{Dom}(\mathfrak{M}); m\rangle \vDash_{\mathcal{L}_{\omega,\omega}} \psi(c)$, for some constant c not in τ and some interpretation of c in the $\{c\}$-structure $\langle \mathrm{Dom}(\mathfrak{M}); m\rangle$.

For example, if $\tau = \{\cdot, <, 1\}$, the vocabulary of Groups, and we let $\mathfrak{M}$ denote the τ-structure $\langle \mathbb{Z}; +, <, 0\rangle$ (the additive Group with the usual ordering on the integers). Then, we will consider whether $\mathfrak{M} \vDash_{\mathcal{L}_{\omega,\omega}} \forall x(\cdot(x, 1) = x)$.

$$
\begin{aligned}
&\mathfrak{M} \vDash_{\mathcal{L}_{\omega,\omega}} \forall x(\cdot(x,1) = x)\\
\Leftrightarrow\ &\mathfrak{M} \vDash_{\mathcal{L}_{\omega,\omega}} \neg\exists x \neg(\cdot(x,1) = x)\\
\Leftrightarrow\ &\mathfrak{M} \nvDash_{\mathcal{L}_{\omega,\omega}} \exists x \neg(\cdot(x,1) = x)\\
\Leftrightarrow\ &\text{there's no } m \in \mathrm{Dom}(\mathfrak{M}) \text{ and constant } c \notin \tau \text{, with } \mathfrak{M} \sqcup \langle \mathfrak{M}; m\rangle \vDash_{\mathcal{L}_{\omega,\omega}} \neg(\cdot(c,1) = c)\\
\Leftrightarrow\ &\text{there's no } m \in \mathrm{Dom}(\mathfrak{M}) \text{ and constant } c \notin \tau \text{, with } \mathfrak{M} \sqcup \langle \mathfrak{M}; m\rangle \nvDash_{\mathcal{L}_{\omega,\omega}} (\cdot(c,1) = c)\\
\Leftrightarrow\ &\text{for any } m \in \mathrm{Dom}(\mathfrak{M}) \text{ and constant } c \notin \tau \text{, } \mathfrak{M} \sqcup \langle \mathfrak{M}; m\rangle \vDash_{\mathcal{L}_{\omega,\omega}} (\cdot(c,1) = c)\\
\Leftrightarrow\ &\text{for any } m \in \mathrm{Dom}(\mathfrak{M}) \text{ and constant } c \notin \tau \text{, } \nu_{\mathfrak{M} \sqcup \langle \mathfrak{M}; m\rangle}(\cdot(c,1) = c) = 1\\
\Leftrightarrow\ &\text{for any } m \in \mathrm{Dom}(\mathfrak{M}) \text{ and constant } c \notin \tau \text{, } \nu^{\star}_{\mathfrak{M} \sqcup \langle \mathfrak{M}; m\rangle}(\cdot(c,1)) = \nu^{\star}_{\mathfrak{M} \sqcup \langle \mathfrak{M}; m\rangle}(c)\\
\Leftrightarrow\ &\text{for any } m \in \mathrm{Dom}(\mathfrak{M}) \text{ and constant } c \notin \tau \text{, } \iota_{\mathfrak{M} \sqcup \langle \mathfrak{M}; m\rangle}(\cdot)(\nu^{\star}_{\mathfrak{M} \sqcup \langle \mathfrak{M}; m\rangle}(c), \nu^{\star}_{\mathfrak{M} \sqcup \langle \mathfrak{M}; m\rangle}(1)) = \iota_{\mathfrak{M} \sqcup \langle \mathfrak{M}; m\rangle}(c)\\
\Leftrightarrow\ &\text{for any } m \in \mathrm{Dom}(\mathfrak{M}) \text{ and constant } c \notin \tau \text{, } \iota_{\mathfrak{M} \sqcup \langle \mathfrak{M}; m\rangle}(c) + \iota_{\mathfrak{M} \sqcup \langle \mathfrak{M}; m\rangle}(1) = m\\
\Leftrightarrow\ &\text{for any } m \in \mathrm{Dom}(\mathfrak{M}) \text{ and constant } c \notin \tau \text{, } m + 0 = m,
\end{aligned}
$$

which is clearly true, and so $\mathfrak{M} \vDash_{\mathcal{L}_{\omega,\omega}} \forall x(\cdot(x,1) = x)$.

We must now verify that First-Order Logic is indeed a Logic.

Theorem 2.4.6. *First-Order Logic is a Logic.*

Sketch of proof:

- Clearly, if $\tau \subseteq \sigma$, then $\mathcal{L}_{\omega,\omega}(\tau) \subseteq \mathcal{L}_{\omega,\omega}(\sigma)$, by our definition;
- if $\mathfrak{M} \vDash_{\mathcal{L}_{\omega,\omega}} \phi$, then, by our definition, $\mathfrak{M}$ is a τ-structure, for some vocabulary τ, and $\phi \in \mathcal{L}_{\omega,\omega}(\tau)$; that is, $\phi \in \mathcal{L}_{\omega,\omega}(\mathrm{Vocab}(\mathfrak{M}))$;
- by our inductive definition of satisfaction, it can be seen that the truth of a sentence depends, fundamentally, only on the truth of the atomic sentences (that is, it is *truth-functional*), and atomic sentences are preserved by isomorphism (that is, if $\mathfrak{M} \cong \mathfrak{N}$, then $\nu_{\mathfrak{M}}(\phi) = \nu_{\mathfrak{N}}(\phi)$) – and so it can be proved, inductively, that the isomorphism property holds;
- again, the reduct property follows by our inductive definition: if $\mathfrak{M} \vDash_{\mathcal{L}_{\omega,\omega}} \phi$, and ϕ is a $\mathcal{L}_{\omega,\omega}(\tau)$-sentence then ϕ depends only and fully on $\mathcal{L}_{\omega,\omega}(\tau)$-atomic sentences, which are described by the τ-reduct of $\mathfrak{M}$, and so the reduct property follows;
- the renaming property also immediately follows from the inductive definitions: the required sentence is just the result of applying the renaming to the original sentence.

■

An interested reader can feel free to work out the intricacies.

2.4.3 Orthodox Logics

In this section, we shall introduce the notion of "Orthodox Logic", of which First-Order Logic one. Orthodox Logics are the main interest of study in Abstract Model Theory (well, really, "Regular Logics" are, but these have slightly more stringent conditions that we shall not go into – all Regular

Logics are Orthodox Logics). Lindström's Theorem, the highlight of this project, is also about Orthodox Logics. Boolean Logics are not, in general, Orthodox, and so in the next section, we shall introduce a class of Logics extending First-Order Logic, which are Orthodox. The properties comprising the definition of "Orthodox Logic", are taken from [Bar16, pp. 29–30]. The theorems in this section exist in "Mathematical Folklore", but the proofs are original.

Here, then, is what we mean when we say a Logic is "Orthodox":

Definition 2.4.7 (Orthodox Logic). Let $\mathcal{L}$ be a Logic. We say that $\mathcal{L}$ is an *Orthodox Logic* if and only if $\mathcal{L}$ has the following properties:

- (*Atom Property*) for all vocabularies τ, and all atomic $\phi \in \mathcal{L}_{\omega,\omega}(\tau)$, there is a sentence $\psi \in \mathcal{L}(\tau)$ such that for any τ-structure $\mathfrak{M}$,
$$\mathfrak{M} \vDash_{\mathcal{L}} \psi \quad \text{if and only if} \quad \mathfrak{M} \vDash_{\mathcal{L}_{\omega,\omega}} \phi;$$
- (*Negation Property*) for all vocabularies τ, and all $\phi \in \mathcal{L}(\tau)$, there is a sentence $\psi \in \mathcal{L}(\tau)$ such that, for any τ-structure $\mathfrak{M}$,
$$\mathfrak{M} \vDash_{\mathcal{L}} \psi \quad \text{if and only if} \quad \mathfrak{M} \nvDash_{\mathcal{L}} \phi;$$
- (*Conjunction Property*) for all vocabularies τ, and all $\phi, \psi \in \mathcal{L}(\tau)$, there is a sentence $\chi \in \mathcal{L}(\tau)$ such that, for any τ-structure $\mathfrak{M}$,
$$\mathfrak{M} \vDash_{\mathcal{L}} \chi \quad \text{if and only if} \quad \mathfrak{M} \vDash_{\mathcal{L}} \phi \text{ and } \mathfrak{M} \vDash_{\mathcal{L}} \psi; \text{ and}$$
- (*Particularisation Property*) for all vocabularies τ, any $c \in \mathrm{Const}(\tau)$, and any $\phi \in \mathcal{L}(\tau)$, there is a sentence $\psi \in \mathcal{L}(\tau \setminus \{c\})$ such that, for any $(\tau \setminus \{c\})$-structure $\mathfrak{M}$,
$$\mathfrak{M} \vDash_{\mathcal{L}} \psi \quad \text{if and only if} \quad \mathfrak{M} \sqcup \langle \mathrm{Dom}(\mathfrak{M}); m\rangle \vDash_{\mathcal{L}} \phi,$$
for some $m \in \mathrm{Dom}(\mathfrak{M})$.

If a Logic has both the Negation Property and the Conjunction Property, we say that it has the *Boolean Property*. If a Logic has the Negation Property, then we use $\neg\phi$ to represent the sentence ψ in the corresponding clause above. If a Logic has the Conjunction Property, then we use $\phi \wedge \psi$ to represent the sentence χ in the corresponding clause above. If a Logic has the Particularisation Property, then we use $\exists c\phi$ to represent the sentence ψ in the corresponding clause above.

We can see that First-Order Logic is an Orthodox Logic quite easily:

Theorem 2.4.8. *First-Order Logic is an Orthodox Logic.*

Proof:

- The Atom Property immediately follows, by definition.
- Let τ be a vocabulary, and let $\phi \in \mathcal{L}_{\omega,\omega}(\tau)$, then note that $\neg\phi \in \mathcal{L}_{\omega,\omega}(\tau)$ and, for any τ-structure $\mathfrak{M}$,
$$\mathfrak{M} \nvDash_{\mathcal{L}_{\omega,\omega}} \phi \quad \text{if and only if } \mathfrak{M} \vDash_{\mathcal{L}_{\omega,\omega}} \neg\phi,$$
by definition. This establishes the Negation Property.

- The proof of the Conjunction Property is similar to that of the Negation Property.
- Let τ be a vocabulary, and let $\phi \in \mathcal{L}_{\omega,\omega}(\tau)$, if there are no constant symbols in τ, we are done, so assume that $c \in \mathrm{Const}(\tau)$. If $\phi \in \mathcal{L}_{\omega,\omega}(\tau \setminus \{c\})$, we are also done, and so we assume that $\phi \notin \mathcal{L}_{\omega,\omega}(\tau \setminus \{c\})$. Then, denote, by ϕ', the result of replacing all occurrences of c in ϕ by v, a variable which does not appear anywhere in ϕ. Then, define ψ to be $\exists v\phi'$. Then, for any $(\tau \setminus \{c\})$-structure $\mathfrak{M}$, we have

  $$\mathfrak{M} \vDash_{\mathcal{L}_{\omega,\omega}} \psi \quad \text{if and only if} \quad \text{there is some } m \in \mathrm{Dom}(\mathfrak{M}) \text{ such that } \mathfrak{M} \sqcup \langle \mathfrak{M}; m \rangle \vDash_{\mathcal{L}_{\omega,\omega}} \phi'(m),$$

  which, obviously, happens if and only if $\mathfrak{M} \sqcup \langle \mathrm{Dom}(\mathfrak{M}); m \rangle \vDash_{\mathcal{L}_{\omega,\omega}} \phi$, for some $m \in \mathrm{Dom}(\mathfrak{M})$. Hence, First-Order Logic has the Particularisation Property.

■

We will implicitly use this Theorem throughout, without making reference to it.

Just as we defined disjunction and implication as abbreviations in First-Order Logic, we can do the same in Logics with the Boolean Property, and, moreover, they will work as expected.

Theorem 2.4.9. *Let $\mathcal{L}$ be a Logic with the Boolean Property, let τ be a vocabulary, and let ϕ and ψ be sentences of $\mathcal{L}(\tau)$. Then, if we define $\phi \vee \psi := \neg(\neg\phi \wedge \neg\psi)$, we see that, for any τ-structure $\mathfrak{M}$,*

$$\mathfrak{M} \vDash_{\mathcal{L}} \phi \vee \psi \quad \textit{if and only if} \quad \mathfrak{M} \vDash_{\mathcal{L}} \phi \textit{ or } \mathfrak{M} \vDash_{\mathcal{L}} \psi.$$

If we define $\phi \to \psi := \neg\phi \vee \psi$, we see that, for any τ-structure $\mathfrak{M}$,

$$\mathfrak{M} \vDash_{\mathcal{L}} \phi \to \psi \quad \textit{if and only if} \quad \mathfrak{M} \vDash_{\mathcal{L}} \phi \textit{ implies } \mathfrak{M} \vDash_{\mathcal{L}} \psi.$$

Proof:

$$\begin{aligned} \mathfrak{M} \vDash_{\mathcal{L}} \phi \vee \psi &\Leftrightarrow \mathfrak{M} \vDash_{\mathcal{L}} \neg(\neg\phi \wedge \neg\psi) \\ &\Leftrightarrow \mathfrak{M} \nvDash_{\mathcal{L}} \neg\phi \wedge \neg\psi \\ &\Leftrightarrow \mathfrak{M} \nvDash_{\mathcal{L}} \neg\phi \text{ or } \mathfrak{M} \nvDash_{\mathcal{L}} \neg\psi \\ &\Leftrightarrow \mathfrak{M} \vDash_{\mathcal{L}} \phi \text{ or } \mathfrak{M} \vDash_{\mathcal{L}} \psi. \end{aligned}$$

And, similarly,

$$\begin{aligned} \mathfrak{M} \vDash_{\mathcal{L}} \phi \to \psi &\Leftrightarrow \mathfrak{M} \vDash_{\mathcal{L}} \neg\phi \vee \psi \\ &\Leftrightarrow \mathfrak{M} \vDash_{\mathcal{L}} \neg\phi \text{ or } \mathfrak{M} \vDash_{\mathcal{L}} \psi \\ &\Leftrightarrow \mathfrak{M} \nvDash_{\mathcal{L}} \phi \text{ or } \mathfrak{M} \vDash_{\mathcal{L}} \psi \\ &\Leftrightarrow \mathfrak{M} \vDash_{\mathcal{L}} \phi \text{ implies } \mathfrak{M} \vDash_{\mathcal{L}} \psi. \end{aligned}$$

■

Henceforth, we shall use the abbreviations "$\phi \vee \psi$" and "$\phi \to \psi$". Moreover, it can be checked by the reader, that if we define $\phi \leftrightarrow \psi := (\phi \to \psi) \wedge (\psi \to \phi)$, then this also acts as expected: it is satisfied if and only if ϕ and ψ both are satisfied, or are both not satisfied. And, so we shall employ this abbreviation also. It can also be verified, that if we define $\forall v\phi := \neg\exists v\neg\phi$, that it acts as expected also.

Thus, we shall make use of this abbreviation too. We will use these abbreviations without reference to the above theorem.

We can see that no Boolean Logic is Orthodox. If a Boolean Logic has the Atom Property and the Particularisation Property, then it cannot have the Particularisation Property, for Boolean Logics, as we have defined them, can only talk about named objects of a structure, never about any arbitrary object of the structure.

We can also have Boolean Logics that do not have the Boolean Property (or, indeed, that do not have Atom Property). For example, our Logic $\mathcal{C}_{D_1,\mu_1}$, from earlier. This does not have the Atom Property, because it has no true sentences; and does not have the Negation Property (*a fortiori*, the Boolean Property) for the same reason. It does, however, have the conjunction property, as no sentence is ever modelled by a structure.

Because $\perp$ is not modelled by any structure, no matter the vocabulary, it follows that $\mathcal{C}_{D_1,\mu_1} \leq \mathcal{L}_{\omega,\omega}$. We can also, using Boolean Logics, define the fragment of First-Order Logic that contains no quantifiers. This is known as Propositional Logic. We will not prove that this is the case, but it easy to see. Let B be any non-degenerate[4] Boolean Algebra. Set $D := \{\top\}$, and let μ be such that $\mu(\phi) = \top$. Then, for any vocabulary τ, as every atomic sentence of $\mathcal{L}_{\omega,\omega}(\tau)$ is an atomic sentence of $\mathbb{B}(\tau)$, with the exception of $\perp$[5], it follows that $\mathcal{B}_{D,\mu}$ is Logically equivalent to the fragment of First-Order Logic without quantifiers. And, consequently, $\mathcal{B}_{D,\mu} \leq \mathcal{L}_{\omega,\omega}$. Moreover, $\mathcal{B}_{D,\mu}$ necessarily has the Boolean Property and the Atom Property.

In general, Abstract Model Theory is concerned with Orthodox Logics, and so henceforth, we shall drop talk of Boolean Algebras, and in our next section, we shall see a new class of Orthodox Logics: Infinitary Logics. Boolean Logics, however, are not without merit, they are used in the field Set Theory for the technique of forcing. In fact, the only Fields Medal for work in Mathematical Logic was awarded to Paul Cohen in 1966, who proved that the Axiom of Choice and generalised continuum hypothesis are independent of the ZF axioms of Set Theory, by considering Boolean Logics (although not quite defined as we have done). For the reader, however, they can provide a rich class of (simple) Logics to play around with.

So prevalent is First-Order Logic in Mathematics (and, in particular, model theory), that we have a special name for when two structures agree on all First-Order sentences:

Definition 2.4.10 (Elementary Equivalence). Let τ be a vocabulary. Let $\mathfrak{M}$ and $\mathfrak{N}$ be τ-structures. We say $\mathfrak{M}$ and $\mathfrak{N}$ are *elementarily equivalent* if for all $\phi \in \mathcal{L}_{\omega,\omega}(\tau)$,

$$\mathfrak{M} \vDash_{\mathcal{L}_{\omega,\omega}} \phi \quad \text{if and only if} \quad \mathfrak{N} \vDash_{\mathcal{L}_{\omega,\omega}} \phi.$$

If $\mathfrak{M}$ and $\mathfrak{N}$ are elementarily equivalent, we write $\mathfrak{M} \equiv \mathfrak{N}$.

Abstract Model Theory is the practice of comparing (Orthodox) Logics: whether that be by Logical strength, properties, or general classifications; we will end with a general classification of First-Order Logic, Lindström's Theorem, by proving that it is the strongest Orthodox Logic to have certain properties.

But, now, let us meet some more Orthodox Logics, Infinitary Logics, which are stronger than First-Order Logic. That is, we shall see that First-Order Logic is a Sublogic of each Infinitary Logic, and

[4] I.e., $\perp \neq \top$.

[5] In $\mathcal{B}_{D,\mu}$, we can simply replace any occurrence of $\perp$ (that is, our First-Order symbol) with $\phi \wedge \neg\phi$ for any sentence ϕ).

that each Infinitary Logic has at least one sentence ϕ (for some vocabulary τ) such that the set of all τ-structures in which ϕ is satisfied is different to any set of τ-structures satisfying any First-Order sentence ψ.

3 Ordinals, Cardinals, and Infinitary Logic

A pre-requisite of reading this project was that the reader had encountered the notion of "uncountability". In this section, we will show the full generalisation of "infinities", and show that there are even different sizes of "uncountable infinities". We will first see an introduction to "ordinal numbers", which are named after the linguistic concept of "ordinal" (English examples are: first, second, third, &c.), which represent numbers associated with orderings. Then, we shall see an introduction to "cardinal numbers", which are named after the linguistic concept of "cardinal" (English examples are: one, two three), which represent numbers associated with counting – or, more specifically, with the size of collections. These two concepts are, in some sense, the foundational results of Set Theory, a field of mathematics which studies structures that satisfy chosen axioms to represent sets, in the same way that Group Theory is the field which studies structures that satisfy the chosen axioms of groups. Finally, we shall use these new concepts to define, and see some examples of, Infinitary Logics, which are (proper) extensions of First-Order Logic.

The sections on Ordinals and Cardinals use definitions from (or, that are strongly based on those given in) [Jec02] and [Kun13].

3.1 Ordinal Numbers

The Ordinal Numbers are generalisation of orderings. Just as we count 0, 1, 2, ..., we may want to continue counting after we have exhausted all the natural numbers. For example, we could have an ordering, where we list all of the natural numbers, and then we list all of the natural numbers again, but with a star after them:

$$0, 1, 2, 3, \dots, 0^{\star}, 1^{\star}, 2^{\star}, \dots$$

By considering ordinals, we can describe the positions of $0^{\star}$, $1^{\star}$, and so on, in the list: $0^{\star}$ is in the ωth position, while 0 is in the 0th position; and, in general, $n^{\star}$ is in the$(\omega + n)$th position.

But what if we had 3 copies, or 4 copies of the natural numbers in a sequence? Or what if we had countably many copies, or uncountably many copies? Ordinals allow us to describe all of these situations.

Ordinals allow us to describe the location of the objects in all of these orders. The class of ordinals, then, is well-ordered; that is, there is a total ordering such that every (non-empty) set of ordinals has a least element. Moreover, every non-empty set of ordinals has a supremum.

The smallest ordinal is 0, and all of the natural numbers are ordinals. We define ω to be the smallest ordinal number bigger than all of the natural numbers. The next ordinal after ω is denoted by $\omega + 1$, after that, $\omega + 2$, and so on. The ordinal ω has no immediate predecessors, and so is called a *limit ordinal*. If an ordinal has an immediate predecessor, it is called a *successor ordinal*.

If α is an ordinal, then there is no ordinal between α and $\alpha + 1$. If α is a limit ordinal, then it is defined to be the supremum of all the preceding ordinals. So, $\omega := \sup\{n \in \mathbb{N}\}$.

We shall use ordinals to index elements of sets: for example, the set $\{x_i;\ i < \alpha\}$ is a set which has elements that have been indexed by the ordinal α.

The smallest limit ordinal after ω is defined as $\sup\{\omega + n;\ n < \omega\} = \omega + \omega = 2\omega$. Returning to our first example, we can define our ordering

$$0, 1, 2, 3, \dots, 0^\star, 1^\star, \dots$$

as follows: let x_i be i if $i < \omega$, otherwise, let x_i be $n^\star$, where $i = \omega + n$; and, so our ordering would be $\{x_i;\ i < \omega\}$.

Using ordinals, we can now define cardinals, using the fact that any well-ordering is order-isomorphic to a set of the form $\{\alpha < \beta;\ \alpha \text{ is an ordinal}\}$, for some ordinal β.

3.2 Cardinal Numbers

We can define the notion of cardinality as follows:

Given a set X, and, assuming the Axiom of Choice, we can well-order X, and so there is an ordinal α such that we can write X as $\{x_i;\ i < \alpha\}$. The least such ordinal is what we call the cardinality of X (and there must be a least one, as the ordinals are well-ordered).

Note, then, that every natural number is a cardinal. For infinite cardinalities, and assuming the Axiom of Choice (which allows us to well-order the cardinals), we write $\aleph_i$, where i is an ordinal. So, $\aleph_0$ is the smallest infinite cardinality (countably infinite), and $\aleph_1$ is the smallest uncountable cardinality.

And, we can call a set X countable if and only if $|X| \leq \aleph_0$, and uncountable otherwise.

Sometimes we write $\aleph_0$ as ω (or ω_0) and $\aleph_1$ as ω_1.

We are now ready to see Infinitary Logics.

3.3 Infinitary Logic

Infinitary Logics are a generalisation of First-Order Logic (and are obviously Orthodox – we shall not prove this fact; we also shall not prove that they are Logics, as they are obviously so: we can easily expand our proofs for First-Order Logic). Infinitary Logics allow for infinite conjunctions and disjunctions of formulæ, rather than our finite conjunctions and disjunctions in First-Order Logic. We define an Infinitary Logic given a cardinal. This cardinal tells us how big of a conjunction or disjunction we can form. Later on, we shall see that Infinitary Logics are more powerful than First-Order Logic, in that they can express more properties of structures (a notion that will be defined more explicitly later on).

We take our definition largely from [Vää11, pp. 157–8]. We write $\mathcal{L}_{\kappa,\omega}$ for the Infinitary Logic which allows conjunctions and disjunctions of less than size κ, for an infinite ordinal $\kappa > \omega$. This is why we write $\mathcal{L}_{\omega,\omega}$ for First-Order Logic, because we are only allowed conjunctions and disjunctions of size less than $\omega = \aleph_0$; i.e., finite conjunctions and disjunctions.

We will now define our formulæ and sentences; our terms and atomic formulæ are the same as in $\mathcal{L}_{\omega,\omega}$.

Definition 3.3.1. Let $\kappa > \aleph_0$ be a cardinal, and τ a vocabulary. Then, the *formulæ of $\mathcal{L}_{\kappa,\omega}$ given the vocabulary* τ is the smallest set containing all of the formulæ of First-Order Logic (given τ) as well as:

- if X is a set of $\mathcal{L}_{\kappa,\omega}(\tau)$-formulæ of cardinality less than κ, then $\bigwedge_{\phi \in X} \phi$ is a $\mathcal{L}_{\kappa,\omega}(\tau)$-formula; and
- if X is a set of $\mathcal{L}_{\kappa,\omega}(\tau)$-formulæ of cardinality less than κ, then $\bigvee_{\phi \in X} \phi$ is a $\mathcal{L}_{\kappa,\omega}(\tau)$-formula.

We can define free and bound variables as in First-Order Logic, and, just like in First-Order Logic, a $\mathcal{L}_{\kappa,\omega}(\tau)$-sentence is a $\mathcal{L}_{\kappa,\omega}(\tau)$-formula with no free variables.

We can now give our definition of satisfaction in Infinitary Logics.

Definition 3.3.2. Given a cardinal $\kappa > \aleph_0$, the Infinitary Logic $\mathcal{L}_{\kappa,\omega}$ is the function $\mathcal{L}_{\kappa,\omega}$, that returns the set of sentences of the Infinitary Logic $\mathcal{L}_{\kappa,\omega}$, given a vocabulary τ, together with the relation $\vDash_{\mathcal{L}_{\kappa,\omega}}$ between structures and $\mathcal{L}_{\kappa,\omega}$-sentences, which is defined, inductively, in the same way as First-Order Logic (for a vocabulary τ, a $\mathcal{L}_{\kappa,\omega}$-sentence ϕ and a τ-structure $\mathfrak{M}$), but with the following additional cases:

- if ϕ is of the form $\bigvee_{\psi \in X} \psi$, then $\mathfrak{M} \vDash_{\mathcal{L}_{\kappa,\omega}} \phi$ if and only if, for some $\psi \in X$, $\mathfrak{M} \vDash_{\mathcal{L}_{\kappa,\omega}} \psi$; and
- if ϕ is of the form $\bigwedge_{\psi \in X} \psi$, then $\mathfrak{M} \vDash_{\mathcal{L}_{\kappa,\omega}} \phi$ if and only if, for every $\psi \in X$, $\mathfrak{M} \vDash_{\mathcal{L}_{\kappa,\omega}} \psi$.

Assuming the Axiom of Choice, consider $\aleph_1$ (which we shall call ω_1 for now), the first uncountable cardinal, and the first cardinal after $\aleph_0$; we will consider the logic $\mathcal{L}_{\omega_1,\omega}$, which allows for countably-infinite conjunctions and disjunctions. Consider $\tau = \{\cdot, <, 1\}$, the language of Ordered Groups, and consider the τ-structure $\langle \mathbb{Z}; +, <, 0\rangle$ (the additive Group with the usual ordering of integers), which we shall denote $\mathfrak{M}$. Define X, a set of $\mathcal{L}_{\omega_1,\omega}(\tau)$-formulæ to be the set, $X := \{x_i;\ 0 \le i < \omega\}$, where each x_i is defined as:

$$x_i := (\overbrace{y \cdot y \cdot \dots \cdot y}^{i+1 \text{ times}}) = 1.$$

Define $\phi := \exists y(\neg(y = 1) \wedge \bigvee_{\psi \in X} \psi(y))$; we will consider whether $\mathfrak{M} \vDash_{\mathcal{L}_{\omega_1,\omega}} \phi$.

$$\mathfrak{M} \vDash_{\mathcal{L}_{\omega_1,\omega}} \phi,$$

if and only if

$$\text{exists } m \in \mathrm{Dom}(\mathfrak{M}) \text{ with } \mathfrak{M} \sqcup \langle \mathrm{Dom}(\mathfrak{M}); m\rangle \vDash_{\mathcal{L}_{\omega_1,\omega}} \neg(c = 1) \text{ and } \mathfrak{M} \sqcup \langle \mathrm{Dom}(\mathfrak{M}); m\rangle \vDash_{\mathcal{L}_{\omega_1,\omega}} \bigvee_{\psi \in X} \psi(c),$$

for some constant $c \notin \tau$, which happens if and only if, for some $\psi \in X$,

$$\mathfrak{M} \sqcup \langle \mathrm{Dom}(\mathfrak{M}); m\rangle \vDash_{\mathcal{L}_{\omega_1,\omega}} \neg(c = 1) \text{ and } \mathfrak{M} \sqcup \langle \mathrm{Dom}(\mathfrak{M}); m\rangle \vDash_{\mathcal{L}_{\omega_1,\omega}} \psi(c),$$

which clearly does not happen, as the only element of the domain satisfying any ψ in X is the identity, but we must choose a non-identity object; hence,

$$\mathfrak{M} \nvDash_{\mathcal{L}_{\omega_1,\omega}} \phi.$$

It can easily be seen that Infinitary Logics are indeed Logics, and, more specifically, are Orthodox Logics: our arguments for First-Order Logic can easily be extended. It is also easy to see that, for each cardinal $\kappa \ge \aleph_0$, $\mathcal{L}_{\omega,\omega} \le \mathcal{L}_{\kappa,\omega}$: just choose the corresponding sentence of $\mathcal{L}_{\omega,\omega}$.

One final note on this section, the second cardinal; i.e., the 'ω' in '$\mathcal{L}_{\kappa,\omega}$' represents the maximum allowed length of a sequence of quantifiers: in our definition, Infinitary Logics, and First-Order Logic only allow finitely many quantifiers in a row in a sentence. However, we could, if desired allow infinitely many, for different cardinalities.

In the next section, we shall see some properties of First-Order Logic, that are of interest. We shall also examine whether Infinitary Logics share these properties.

4 Properties of Logics

Before we continue, we shall quickly meet two definitions, which we shall use frequently from here on.

Definition 4.0.1. Let $\mathcal{L}$ be a Logic, and τ a vocabulary. We say that a sentence $\phi \in \mathcal{L}(\tau)$ is *satisfiable* if and only if there exists a τ-structure $\mathfrak{M}$ such that

$$\mathfrak{M} \vDash_{\mathcal{L}} \phi.$$

If ϕ is not satisfiable, we say that ϕ is *unsatisfiable*.

We can generalise Definition 4.0.1, to deal with satisfiability of sets.

Definition 4.0.2. We say that a set of sentences $X \subseteq \mathcal{L}(\tau)$ is *satisfiable* if and only if there exists a τ-structure $\mathfrak{M}$ such that, for each $\phi \in X$,

$$\mathfrak{M} \vDash_{\mathcal{L}} \phi.$$

And, if X is not satisfiable, we say that X is *unsatisfiable*.

We are now ready to proceed, and in this section, we shall engage more with Abstract Model Theory: we will define some properties of Logics, and prove that First-Order Logic satisfies these properties. The first is Compactness, which tells us that to prove that a set of sentences of a given Logic (given a vocabulary) is satisfiable, we only need to prove that every finite subset of the set is satisfiable. The second is the Löwenheim-Skolem-Tarski Property, which tells us that if a sentence of a given Logic ϕ (given a vocabulary τ) is true in a τ-structure with an infinite domain, then it is possible to find a τ-structure whose domain has cardinality κ (for any infinite cardinal κ) and such that ϕ is also true (according to the Logic in question) in that τ-structure.

A possible project, then, in Abstract Model Theory, is to see which Logics have these properties, and perhaps, even prove a general statement providing sufficient and necessary conditions for Logics to have these properties.

We shall undertake a slightly different project: we shall see that Lindström's Theorem characterises First-Order Logic as the strongest Logic (in terms of expressibility; which is defined in the next section) which has both the Compactness and Löwenheim-Skolem-Tarski Properties (in fact, we prove a stronger result, as we prove it for weaker versions of Compactness and Löwenheim-Skolem-Tarski, both of which we define in this section).

We shall also examine Infinitary Logics in this section, in relation to Compactness and Löwenheim-Skolem-Tarski. I take the definitions in this section from [Ebb16, pp. 31–32], and proofs are attributed as appropriate.

4.1 Compactness

Definition 4.1.1. Let $\mathcal{L}$ be a logic. We say $\mathcal{L}$ has the *Compactness Property* if and only if for any vocabulary τ we have that a set of sentences $X \subseteq \mathcal{L}(\tau)$ is satisfiable if and only if every finite subset of X is satisfiable.

Theorem 4.1.2. *First-Order Logic has the Compactness Property.*

Proof: adapted from PY4612 Advanced Logic. We assume the Soundness and Completeness Theorem of First-Order Logic[6], which says that there is a proof-system for First-Order Logic so that $X \vDash_{\mathcal{L}_{\omega,\omega}} \phi$[7] if and only if there is a finite proof from X to ϕ (where the details of "finite proof" are omitted here). So, ϕ can be proved in our finite proof system, from X if and only if it can be proved from a finite subset of X. It then follows, by Soundness and Completeness, that there exists a finite subset X' of X such that $X \vDash_{\mathcal{L}_{\omega,\omega}} \phi$ if and only if $X' \vDash_{\mathcal{L}_{\omega,\omega}} \phi$.

By our definition of satisfiability, we know that $Y \subseteq \mathcal{L}_{\omega,\omega}(\tau)$ is satisfiable if and only if there is a τ-structure $\mathfrak{M}$ such that $\mathfrak{M} \vDash_{\mathcal{L}_{\omega,\omega}} Y$, but then, it is clear that $Y \nvDash_{\mathcal{L}_{\omega,\omega}} \perp$, as $\mathfrak{M} \nvDash_{\mathcal{L}_{\omega,\omega}} \perp$, by definition. So, by the above, there is a finite subset X' of X such that $X \vDash_{\mathcal{L}_{\omega,\omega}} \perp$ if and only if $X' \vDash_{\mathcal{L}_{\omega,\omega}} \perp$. I.e., there is a finite subset X' of X such that X is satisfiable if and only if X' is satisfiable. *a fortiori*, if every finite subset of X is satisfiable, then so is X.

Clearly, $X \vDash_{\mathcal{L}_{\omega,\omega}} X'$ for any finite subset X' of X, and so if X is satisfiable, so is every finite subset of X. □

We can define more fine-grained versions of Compactness:

Definition 4.1.3. Let $\mathcal{L}$ be a logic. Given a cardinal κ, we say that $\mathcal{L}$ has the *κ-Compactness Property* if and only if for any vocabulary τ, we have that a set of sentences $X \subseteq \mathcal{L}(\tau)$, where $|X| \leq \kappa$, is satisfiable if and only if every finite subset of X is satisfiable.

Sometimes, instead of saying $\aleph_0$-Compactness, we say ω-Compactness, or Countable Compactness. It is clear that a Logic has the Compactness Property if and only if, for any infinite cardinal κ, the given Logic has the κ-Compactness Property. And, so we deduce that First-Order Logic is κ-Compact (that is, has the κ-Compactness Property) for any infinite cardinal κ.

A natural question one might ask, given our new definition, from the perspective of an Abstract Model Theorist, is whether it is true that every Orthodox Logic has the Compactness Property (or κ-Compactness for some cardinal κ). We can show that it is not the case that every Orthodox Logic has

[6] See PY4612 Advanced Logic for a proof.

[7] This notation means that if $X \cup \{\phi\} \subseteq \mathcal{L}_{\omega,\omega}(\tau)$, then in every τ-structure $\mathfrak{M}$ such that $\mathfrak{M} \vDash_{\mathcal{L}_{\omega,\omega}} X$, we also have $\mathfrak{M} \vDash_{\mathcal{L}_{\omega,\omega}} \phi$.

the ω-Compactness Property (and consequently the Compactness Property), with a counter-example: Infinitary Logic. We can see that ω-Compactness fails for any Infinitary Logic.

Theorem 4.1.4 ([Hod97, p. 127]). *No Infinitary Logic has the ω-Compactness Property.*

Proof: let κ be such that $\kappa > \aleph_0$, and let $\tau = \{c_i;\ 0 \leq i < \omega\}$, where each c_i is a constant symbol. Then, consider $X \subseteq \mathcal{L}_{\kappa,\omega}(\tau)$ defined as

$$X := \{\neg(c_0 = c_1), \neg(c_0 = c_2), \neg(c_0 = c_3), \ldots\}.$$

If we add another sentence to X to obtain $X' := X \cup \{\bigvee_{0<i<\omega} c_0 = c_i\}$, then we can see that any finite subset Γ of X' is satisfiable: define $C := \{c_i;\ c_i \neq c_0$ and c_i appears in a sentence of $\Gamma\}$, then any τ-structure which interprets c_0 to a fixed element of the domain (say m) and maps each $c_i \in C$ to a distinct element of the domain (excluding m) satisfies Γ; and this is possible if the structure has an infinite domain, so we conclude that every finite subset of X' is satisfiable.

But, we can see that X' is obviously not satisfiable, hence ω-Compactness for $\mathcal{L}_{\kappa,\omega}$ must fail, for otherwise, we would be able to conclude that X' is satisfiable. $\square$

It follows that no Infinitary Logic has the Compactness Property. This is interesting, because it shows us that just by adding infinitely long disjunctions to First-Order Logic, we would have to give up the Compactness Property.

4.2 Löwenheim-Skolem-Tarski

The Compactness Property tells us something about sets of sentences in a Logic; our next result tells us something about the structures, that satisfy certain sentences, themselves (although this is still in regards to the Logic, for satisfaction is Logic-relative).

Definition 4.2.1. Let $\mathcal{L}$ be a Logic, and κ an infinite cardinal. We say that $\mathcal{L}$ has the *κ-Downward-Löwenheim-Skolem-Tarski Property* if and only if for any (countable[8]) vocabulary τ, we have that for any $\phi \in \mathcal{L}(\tau)$ that if there exists a τ-structure $\mathfrak{M}$ such that $\mathfrak{M} \vDash_{\mathcal{L}} \phi$ and $|\mathrm{Dom}(\mathfrak{M})| \geq \kappa$, then there is a τ-structure $\mathfrak{N}$ such that $\mathfrak{N} \vDash_{\mathcal{L}} \phi$, and $\aleph_0 \leq |\mathrm{Dom}(\mathfrak{M})| \leq \kappa$.

Definition 4.2.2. Let $\mathcal{L}$ be a Logic, and κ an infinite cardinal. We say that $\mathcal{L}$ has the *κ-Upward-Löwenheim-Skolem-Tarski Property* if and only if for any (countable[9]) vocabulary τ, we have that for any $\phi \in \mathcal{L}(\tau)$ that if there exists a τ-structure $\mathfrak{M}$ such that $\mathfrak{M} \vDash_{\mathcal{L}} \phi$ and $\aleph_0 \leq |\mathrm{Dom}(\mathfrak{M})| \leq \kappa$, then there is a τ-structure $\mathfrak{N}$ such that $\mathfrak{N} \vDash_{\mathcal{L}} \phi$, and $|\mathrm{Dom}(\mathfrak{M})| \geq \kappa$.

Again, we have generalised versions of these definitions:

Definition 4.2.3. Let $\mathcal{L}$ be a Logic. Then, we say that $\mathcal{L}$ has the *Downward-Löwenheim-Skolem-Tarski Property* if and only if, for all infinite cardinals κ, $\mathcal{L}$ has the κ-Downward-Löwenheim-Skolem-Tarski Property.

[8] Our definition relies heavily on the vocabulary being countable.
[9] Again, this is important.

Definition 4.2.4. Let $\mathcal{L}$ be a Logic. Then, we say that $\mathcal{L}$ has the *Upward-Löwenheim-Skolem-Tarski Property* if and only if, for all infinite cardinals κ, $\mathcal{L}$ has the κ-Upward-Löwenheim-Skolem-Tarski Property.

Theorem 4.2.5. *First-Order Logic has the Downward Löwenheim-Skolem-Tarski Property.*

Sketch of proof: see PY4612 Advanced Logic, where this is proved by means of a construction used in the proof of Completeness. In fact, what is proved is that First-Order Logic has the $\aleph_0$-Downward-Löwenheim-Skolem-Tarski Property. But, this (obviously) implies the stronger claim that First-Order Logic has the Downward-Löwenheim-Skolem-Tarski Property. □

And, so, for any infinite cardinal κ, First-Order Logic has the κ-Downward-Löwenheim-Skolem-Tarski Property. It is also true that First-Order Logic has the Upward-Löwenheim-Skolem-Tarski Property.

Theorem 4.2.6. *First-Order Logic has the Upward-Löwenheim-Skolem-Tarski Property.*

Sketch of proof: (based on the proof given in [Hod97, p. 127]). Let κ be an infinite cardinal, and τ a (countable) vocabulary. Let $\phi \in \mathcal{L}_{\omega,\omega}(\tau)$ be such that there exists a τ-structure $\mathfrak{M}$ with $\mathfrak{M} \vDash_{\mathcal{L}_{\omega,\omega}} \phi$, and $\aleph_0 \leq |\mathrm{Dom}(\mathfrak{M})| \leq \kappa$.

We expand the vocabulary τ be adding κ-"many" new constants to τ, and call the resulting vocabulary σ: that is, $\tau \subseteq \sigma$ and $|\mathrm{Const}(\sigma \setminus \tau)| = \kappa$. Note that σ is not necessarily countable (σ is countable if and only if $\kappa = \aleph_0$).

Then, we define a new set of sentences $X \subseteq \mathcal{L}_{\omega,\omega}(\sigma)$:

$$X := \{\neg(c_1 = c_2);\ c_1 \neq c_2 \text{ and } c_1, c_2 \in \mathrm{Const}(\sigma \setminus \tau)\}.$$

We then show that the set of $\mathcal{L}_{\omega,\omega}(\sigma)$-sentences $\{\phi\} \cup X$ is satisfiable, using the Compactness Property of First-Order Logic. That is, we show that every finite subset of $\{\phi\} \cup X$ is satisfiable, and then it follows that the entire set is satisfiable.

To see that every finite subset is satisfiable, note that any finite subset X' of X contains at most finitely many constant symbols, then, we can expand our τ-structure $\mathfrak{M}$ into a σ-structure, by interpreting each of the finitely many constant symbols in X' as distinct elements, and all the (infinitely many) others that do not appear in our finite subset X' of X to all be the same element of $\mathrm{Dom}(\mathfrak{M})$. Clearly, the expanded structure models ϕ (as it did originally), and models X', by construction. Hence, every finite subset of $\{\phi\} \cup X$ is satisfiable, and thus there is a σ-structure $\mathfrak{N}$ such that $\mathfrak{N} \vDash_{\mathcal{L}_{\omega,\omega}} \{\phi\} \cup X$, by the Compactness Property of First-Order Logic.

It then follows that the τ-reduct, $\mathfrak{N} \upharpoonright \tau$ of $\mathfrak{N}$ is a τ-structure, with $|\mathrm{Dom}(\mathfrak{N} \upharpoonright \tau)| \geq \kappa$, as $\mathrm{Dom}(\mathfrak{N}) = \mathrm{Dom}(\mathfrak{N} \upharpoonright \tau)$, and because $\mathfrak{N} \vDash X$, which clearly forces a domain of at least κ elements. Moreover, because of the reduct property of Logics, $\mathfrak{N} \upharpoonright \tau \vDash_{\mathcal{L}_{\omega,\omega}} \phi$ as $\mathfrak{N} \vDash_{\mathcal{L}_{\omega,\omega}} \phi$, by construction.

Hence, $\mathfrak{N} \upharpoonright \tau$ satisfies the requirements of the theorem. □

And hence, First-Order Logic has both properties.

Corollary 4.2.7. *First-Order Logic has both the Downward and the Upward Löwenheim-Skolem-Tarski Properties.*

Proof: This follows immediately from Theorem 4.2.5 and Theorem 4.2.6. □

This tells us that if a sentence ϕ of $\mathcal{L}_{\omega,\omega}(\tau)$ is true in some τ-structure with an infinite domain, then there exists a τ-structure, for any infinite cardinal κ, such that ϕ is true in it, and it has a domain with cardinality κ.

Again, returning to comparisons with Infinitary Logic, there are versions of the Downward Löwenheim-Skolem-Tarski Property that hold for Infinitary Logics; unfortunately, we would need to do a lot more work with Infinitary Logics to even state, let alone prove such a result. Details can be found in [Mar16, pp. 11–12].

We can, however, show that the Upward Löwenheim-Skolem-Tarski Theorem fails for Infinitary Logics. First, we show that there is a satisfiable sentence (in a specific vocabulary) of Infinitary Logics that forces any model have a countably-infinite domain.

Lemma 4.2.8. *Let $\mathcal{L}$ be an Infinitary Logic, and τ be a vocabulary, consisting of countably-infinitely many constant symbols. Then there is a satisfiable sentence $\psi \in \mathcal{L}(\tau)$ such that if $\mathfrak{M}$ is a τ-structure, then $\mathfrak{M} \vDash_{\mathcal{L}} \psi$ implies $|Dom(\mathfrak{M})| = \aleph_0$.*

Proof: let τ be a vocabulary consisting of countably-infinitely many constant symbols c_i, $0 < i < \omega$. Then define

$$\phi := \forall x\Big(\bigvee_{0<i<\omega} (x = c_i)\Big) \wedge \bigwedge_{0<i<\omega} D_i,$$

where each D_i is defined in the following way:

$$D_i := \bigwedge_{0<j<\omega; j\neq i} \neg(c_i = c_j).$$

The left conjunct (of ϕ) expresses the fact that every element of the domain is interpreted onto by one of the constant symbols of τ, and the right conjunct (of ϕ) expresses the fact that no two constant symbols are interpreted as the same element of the domain. Hence, any τ-structure that satisfies ϕ is such that the interpretation of all the constant symbols map to distinct elements of the domain, and every element of the domain is interpreted by some constant symbol of τ, of which there are countably-infinite.

This is clearly a possible structure, and so there are models of ϕ, and, from the above, the domain of any model of ϕ must be countably-infinite. ■

Now, since the domain of any model of ϕ must be countably-infinite, we must also have that Upward-Löwenheim-Skolem-Tarski does not hold for any Infinitary Logic, as it would imply that we could find models of ϕ with cardinality greater than $\aleph_0$, but we cannot. In particular, then, for any cardinal $\kappa > \aleph_0$, Infinitary Logics do not have the κ-Upward-Löwenheim-Skolem-Tarski Property.

5 Expressibility and Isomorphisms

This section collates many results which will be useful in our proof of Lindström's Theorem, in the next section. In the first subsection, we also introduce the concept of "expressibility": we say that a property of structures is expressible in a logic $\mathcal{L}$ (given a vocabulary τ) if and only if there is a sentence of $\mathcal{L}(\tau)$ which is modelled in only and all those τ-structures with the property in question. For example, the property of "being a structure" is expressible in First-Order Logic (given any vocabulary), because $\neg\bot$ is true in every structure, and every structure (and only structures) are structures. Another example, given the vocabulary of Groups τ, the property of "being a Group" is expressible in First-Order Logic, because any τ-structure satisfying the Group axioms (as standardly formalised in First-Order Logic) is a Group (and only such structures are Groups). In the second subsection, we prove some results about structures, isomorphisms, and partial isomorphisms, which will be useful in our final section on Lindström's Theorem.

5.1 Expressibility

Having seen some properties that are expressible in First-Order Logic, we can now show that some properties are not expressible. We can use the Compactness and Löwenheim-Skolem-Tarski Properties of First-Order Logic to this end. Our examples here shall be about cardinality.

Our first example of a property of structures that is inexpressible in First-Order Logic is that of "being a finite structure". We shall show that this is in expressible, by using the Compactness Property.

Theorem 5.1.1. *There is no vocabulary τ such that there exists a $\phi \in \mathcal{L}_{\omega,\omega}(\tau)$ such that $\mathfrak{M} \vDash_{\mathcal{L}_{\omega,\omega}} \phi$ if and only if $|Dom(\mathfrak{M})| < \aleph_0$.*[10]

Proof: suppose otherwise, that ϕ is such a sentence for some vocabulary τ. Then consider the following set $X \subseteq \mathcal{L}_{\omega,\omega}(\tau)$:

$$X := \{\phi, \neg\exists y_1 \forall x(y_1 = x), \neg\exists y_1 \exists y_2 \forall x(\neg(y_1 \neq y_2) \wedge (y_1 = x \vee y_2 = x)), \ldots\}.$$

So that the first sentence after ϕ expresses the fact that there is not precisely one thing in the domain, the second sentence after ϕ expresses the fact that there is not precisely two things in the domain, and so on.

We can see that any finite subset of X is satisfiable: take any τ-structure which has a finite domain bigger than which any sentence of the finite subset prevents. Hence, by the Compactness Property (Countable Compactness, in particular), we see that all of X is satisfiable: let $\mathfrak{M}$ be such a τ-structure. But, as every sentence in X must be true in $\mathfrak{M}$, which means that $|\mathrm{Dom}(\mathfrak{M})| \neq 1$, $|\mathrm{Dom}(\mathfrak{M})| \neq 2$, &c., by each sentence of X that is not ϕ. But, as ϕ also is true in $\mathfrak{M}$, by construction, the domain of $\mathfrak{M}$ is finite, but there is no possible choice for this (remember, we disallow structures with an empty domain), hence we must conclude that no such sentence ϕ can exist. □

However, when we move to Infinitary Logics, we can express such a property of structures (and, in fact, there is such a sentence, irregardless of the vocabulary).

Theorem 5.1.2. *If $\mathcal{L}$ is an Infinitary Logic, and τ a vocabulary, then there is a sentence $\psi \in \mathcal{L}(\tau)$ such that if $\mathfrak{M}$ is a τ-structure, then $\mathfrak{M} \vDash_{\mathcal{L}} \psi$ if and only if $|Dom(\mathfrak{M})| < \aleph_0$.*

[10] This is an example from PY4612.

Proof: let ψ be

$$\psi := \bigvee_{0<i<\omega} C_i,$$

where each C_i expresses that there are precisely n elements (these are the un-negated versions of the sentences in X in Theorem 5.1.1, other than ϕ, of course).

Clearly, then, this expresses that there is precisely 1 element in the domain, or there are precisely 2 elements in the domain, or 3, and so on.

Hence, there must be a finite number of elements in the domain of any τ-structure satisfying ψ. ■

This is one case in which we can say that Infinitary Logics are more expressive than First-Order Logic.

We know, from our proofs of Theorem 5.1.1 and Theorem 5.1.2 that First-Order Logic can express that a structure has precisely n elements, for any finite $n > 0$. However, we know, from Theorem 5.1.1 that we cannot express that a structure is finite, without reference to specific cardinality, in First-Order Logic. As First-Order Logic is an Orthodox Logic, it follows that there is no sentence expressing that a structure is not-finite either (for if there was, then the negation of "not-finite" would be "finite"). That is, there is no sentence true in only and all those structure with an infinite domain. We shall see, in our next example that, in First-Order Logic, we also cannot express that the domain of the structure is countably infinite (or in fact, that it has cardinality κ for any infinite cardinal κ). And, as First-Order Logic is an Orthodox Logic, we also cannot express the fact that a structure does not have cardinality κ for any infinite cardinal κ.

Theorem 5.1.3. *let κ be an infinite cardinal, and τ a vocabulary. Then there is no sentence $\phi \in \mathcal{L}_{\omega,\omega}(\tau)$ such that, for any τ-structure $\mathfrak{M}$,*

$$\mathfrak{M} \vDash_{\mathcal{L}_{\omega,\omega}} \phi \quad \textit{if and only if} \quad |Dom(\mathfrak{M})| = \kappa.$$

Proof: Let κ be an infinite cardinal, and τ a vocabulary. Then, if $\phi \in \mathcal{L}_{\omega,\omega}(\tau)$ were to be such that for any τ-structure $\mathfrak{M}$,

$$\mathfrak{M} \vDash_{\mathcal{L}_{\omega,\omega}} \phi \quad \text{if and only if} \quad |\mathrm{Dom}(\mathfrak{M})| = \kappa,$$

then, as there certainly are τ-structures with domain of size κ, any of these structures must satisfy ϕ. But then, by the Löwenheim-Skolem-Tarski Property of First-Order Logic, we can find another structure $\mathfrak{N}$, with a domain of cardinality λ, where $\lambda \neq \kappa$ and $\lambda \geq \aleph_0$, and $\mathfrak{N} \vDash_{\mathcal{L}_{\omega,\omega}} \phi$, contradicting our assumption. ■

Seeing what properties a Logic can express (given certain vocabularies) is one method of categorising them (although a very coarse method), and so, this is, in some sense, a part of Abstract Model Theory.

5.1.1 Abstract Model Theory

In this part, we will do some "real Abstract Model Theory", by showing facts about all Orthodox Logics (which are typically the subject of study in Abstract Model Theory) that extend First-Order Logic. The proofs, and their statements, in this part are all reconstructed from the proof of Lindström's Theorem given in [Flu16] (many of which are implicitly assumed) – this continues until part

5.2.1. We begin by showing that if there is a sentence that First-Order Logic cannot express in an Orthodox Logic, then there must be at least one structure in which that sentence is true. To do that, we begin by defining these notions formally.

Definition 5.1.4. Let $\mathcal{L} \geq \mathcal{L}_{\omega,\omega}$ be a Logic, τ a vocabulary, and $\psi \in \mathcal{L}(\tau)$. We say ψ is *not equivalent to any First-Order sentence* if there is no $\phi \in \mathcal{L}_{\omega,\omega}(\tau)$, such that for all τ-structures $\mathfrak{M}$,

$$\mathfrak{M} \vDash_{\mathcal{L}} \psi \quad \text{if and only if} \quad \mathfrak{M} \vDash_{\mathcal{L}} \phi.$$

Now, we will show that any sentence of an Orthodox Logic, which is strictly stronger than First-Order Logic, that is not equivalent to a First-Order sentence, must be satisfiable.

Theorem 5.1.5. *Let $\mathcal{L} \geq \mathcal{L}_{\omega,\omega}$ be an Orthodox Logic, and τ a vocabulary. Then, if $\psi \in \mathcal{L}(\tau)$ is not equivalent to any First-Order sentence, then ψ is satisfiable.*

Proof: suppose otherwise: that ψ is not satisfiable. Then, for every τ-structure $\mathfrak{M}$, we have

$$\mathfrak{M} \nvDash_{\mathcal{L}} \psi;$$

however, we also have

$$\mathfrak{M} \nvDash_{\mathcal{L}} \perp.$$

Hence, for every τ-structure $\mathfrak{M}$,

$$\mathfrak{M} \vDash_{\mathcal{L}} \psi \quad \text{if and only if} \quad \mathfrak{M} \vDash_{\mathcal{L}} \perp,$$

which contradicts our assumption that ψ was not equivalent to any First-Order sentence. Thus, we conclude that ψ is satisfiable. ■

We have now done our first bit of Abstract Model Theory: we have proved a result about all Orthodox Logics stronger than First-Order Logic (and in fact all those sentences which make it stronger than First-Order Logic). This is a very general statement, and so is truly part of Abstract Model Theory. We shall continue in this way, proving some results about these kinds of logic, in preparation for the big "foundational" result of Abstract Model Theory: Lindström's Theorem.

Theorem 5.1.6. *Let $\mathcal{L} \geq \mathcal{L}_{\omega,\omega}$ be an Orthodox Logic, τ a vocabulary, $\psi \in \mathcal{L}(\tau)$ be not equivalent to any First-Order sentence, and $\phi \in \mathcal{L}_{\omega,\omega}(\tau)$. Then, either $\psi \wedge \phi$ or $\psi \wedge \neg\phi$ is not equivalent to a First-Order sentence.*

Proof: suppose $\psi \wedge \phi$ is equivalent to $\chi \in \mathcal{L}_{\omega,\omega}(\tau)$. Then, for each τ-structure $\mathfrak{M}$,

$$\mathfrak{M} \vDash_{\mathcal{L}} \psi \wedge \phi \quad \text{if and only if} \quad \mathfrak{M} \vDash_{\mathcal{L}} \chi.$$

Suppose also that $\psi \wedge \neg\phi$ is equivalent to $\chi' \in \mathcal{L}_{\omega,\omega}(\tau)$. So, for each τ-structure $\mathfrak{N}$,

$$\mathfrak{N} \vDash_{\mathcal{L}} \psi \wedge \neg\phi \quad \text{if and only if} \quad \mathfrak{N} \vDash_{\mathcal{L}} \chi'.$$

Let $\mathfrak{O}$ be a τ-structure. Then, by how disjunction and conjunction behave in Orthodox Logics, we have

$$\mathfrak{O} \vDash_{\mathcal{L}} \chi \vee \chi' \Rightarrow \mathfrak{O} \vDash_{\mathcal{L}} \psi.$$

Suppose that $\mathfrak{O} \vDash_{\mathcal{L}} \psi$. Then, as $\phi \in \mathcal{L}_{\omega,\omega}(\tau)$, we must have $\mathfrak{O} \vDash_{\mathcal{L}} \phi$ or $\mathfrak{O} \vDash_{\mathcal{L}} \neg\phi$. In either case, we can conclude $\mathfrak{O} \vDash_{\mathcal{L}} \chi \vee \chi'$, as $\mathcal{L}$ is an Orthodox Logic.

Thus, we have shown that ψ is equivalent to $\chi \vee \chi' \in \mathcal{L}_{\omega,\omega}(\tau)$, which is a contradiction. So, we conclude that we cannot have both $\psi \wedge \phi$ and $\psi \wedge \neg\phi$ equivalent to a First-Order sentence. ∎

In Theorem 5.1.6, we cannot remove the 'or' condition. That is, we cannot replace the final sentence with 'Then, $\psi \wedge \phi$ is not equivalent to a First-Order sentence'. To see this, note that after one application, of Theorem 5.1.6, we have a sentence $\psi \wedge \phi$ such that it is not equivalent to a First-Order sentence, but then, by taking $\phi' := \neg\phi$, we could apply Theorem 5.1.6 once again to conclude that $\psi \wedge \phi \wedge \phi'$ is not equivalent to any First-Order sentence. However, it is obvious that it can never be satisfied, contradicting the fact that if a sentence is not equivalent to any First-Order sentence, it is satisfiable (Theorem 5.1.5).

Also, we cannot replace the final sentence with 'Then, $\psi \wedge \neg\phi$ is not equivalent to a First-Order sentence', as we can run the same argument with 'ϕ' replaced by '$\neg\phi$'. However, we can glean a little more information: if the First-Order sentence does not 'interfere' with the non-First-Order sentence, then we can conclude that the First-Order sentence also does not 'interfere' with the negation of the non-First-Order sentence. This is proved next.

Theorem 5.1.7. *Let τ be a vocabulary, and let $\mathcal{L} \geq \mathcal{L}_{\omega,\omega}$ be an Orthodox Logic. Then, let $\psi \in \mathcal{L}(\tau)$ be not equivalent to any First-Order sentence, and $\phi \in \mathcal{L}_{\omega,\omega}(\tau)$ be satisfiable. Then, if $\psi \wedge \phi$ is not equivalent to any First-Order sentence, nor is $\neg\psi \wedge \phi$.*

Proof: assume the contrary; so, there exists $\chi \in \mathcal{L}_{\omega,\omega}(\tau)$ such that for any τ-structure $\mathfrak{M}$, we have $\mathfrak{M} \vDash_{\mathcal{L}} \neg\psi \wedge \phi$ if and only if $\mathfrak{M} \vDash_{\mathcal{L}} \chi$. Now, let $\mathfrak{M}$ be an arbitrary τ-structure. Then, if we have $\mathfrak{M} \vDash \phi \wedge \neg\chi$, we must have, as $\mathcal{L}$ is Orthodox, $\mathfrak{M} \vDash_{\mathcal{L}} \phi \wedge \neg(\neg\psi \wedge \phi)$; i.e., $\mathfrak{M} \vDash_{\mathcal{L}} \psi \wedge \phi$.

Suppose $\mathfrak{M} \vDash_{\mathcal{L}} \psi \wedge \phi$, where $\mathfrak{M}$ is a τ-structure. Then, we also have, as $\mathcal{L}$ is Orthodox, $\mathfrak{M} \vDash_{\mathcal{L}} \phi \wedge (\psi \vee \neg\phi)$; i.e., $\mathfrak{M} \vDash_{\mathcal{L}} \phi \wedge \neg\chi$.

It is clear that $\phi \wedge \neg\chi$ is a First-Order sentence, as both ϕ and $\neg\chi$ are. So, we have shown that $\psi \wedge \phi$ is equivalent to a First-Order sentence. Hence, our assumption must be false. Thus, we conclude that if $\psi \wedge \phi$ is not equivalent to any First-Order sentence, nor is $\neg\psi \wedge \phi$. ∎

We can generalise Theorem 5.1.6.

Theorem 5.1.8. *Let $\mathcal{L} \geq \mathcal{L}_{\omega,\omega}$ be an Orthodox Logic, τ a vocabulary, $X \subseteq \mathcal{L}(\tau)$ be satisfiable, and $\phi \in \mathcal{L}_{\omega,\omega}(\tau)$ be satisfiable. Then, either $X \cup \{\phi\}$ or $X \cup \{\neg\phi\}$ is satisfiable.*

Proof: suppose that $X \cup \{\phi\}$ is unsatisfiable. Then, choose a τ-structure $\mathfrak{M}$ such that $\mathfrak{M} \vDash_{\mathcal{L}} \phi$. We can do this because we supposed that ϕ was satisfiable.

But, because we supposed that $X \cup \{\phi\}$ was unsatisfiable, we must have that $\mathfrak{M} \nvDash_{\mathcal{L}} X$, as otherwise $X \cup \{\phi\}$ would be satisfied by $\mathfrak{M}$.

As $\mathfrak{M}$ was arbitrarily chosen, we can conclude that for any τ-structures $\mathfrak{M}$,

$$\text{if} \quad \mathfrak{M} \vDash_{\mathcal{L}} \phi, \quad \text{then} \quad \mathfrak{M} \nvDash_{\mathcal{L}} X.$$

And, so, by contraposition,

$$\begin{aligned}\mathfrak{M} \models_{\mathcal{L}} X &\Rightarrow \mathfrak{M} \not\models_{\mathcal{L}} \phi \\ &\Rightarrow \mathfrak{M} \models_{\mathcal{L}} \neg\phi, \quad \text{as } \mathcal{L} \text{ is Orthodox.}\end{aligned}$$

But, if, for some τ-structure $\mathfrak{M}$, we have $\mathfrak{M} \models_{\mathcal{L}} X$ and $\mathfrak{M} \models_{\mathcal{L}} \neg\phi$, then we have shown that $X \cup \{\neg\phi\}$ is satisfiable.

On the other hand, if we assume there is no τ-structure $\mathfrak{M}$ such that $\mathfrak{M} \models_{\mathcal{L}} X$ and $\mathfrak{M} \models_{\mathcal{L}} \neg\phi$ (i.e., $X \cup \{\neg\phi\}$ is unsatisfiable), then because X is satisfiable, we have contradicted the claim that for any τ-structure $\mathfrak{M}$,

$$\text{if} \quad \mathfrak{M} \models_{\mathcal{L}} X, \quad \text{then} \quad \mathfrak{M} \models_{\mathcal{L}} \neg\phi.$$

Hence, we must deny our assumption that $X \cup \{\phi\}$ is unsatisfiable. Thus, either $X \cup \{\phi\}$ or $X \cup \{\neg\phi\}$ must be satisfiable. ■

As with Theorem 5.1.6, we also cannot drop the 'or' condition in Theorem 5.1.8. It is for the same reason, as we saw for Theorem 5.1.6. That is, we cannot replace the final sentence of the statement of Theorem 5.1.8, by 'Then, $X \cup \{\phi\}$ is satisfiable'. For, by one application of Theorem 5.1.8, we conclude that $X \cup \{\phi\}$ is satisfiable. But, we could also use Theorem 5.1.8 to conclude that $X \cup \{\phi\} \cup \{\neg\phi\}$ is satisfiable. This is clearly false. Again, the same argument runs through with 'ϕ' replaced by '$\neg\phi$'.

We now have enough information about Orthodox Logics that extend First-Order Logic for Lindström's Theorem. But, before we can prove it, we need to see some more information about Isomorphisms, and the Back-and-Forth method.

5.2 Isomorphisms and the Back-and-Forth Method

Now, we shall see that in finite structures of a finite vocabulary, we can encode sufficient information into a single First-Order sentence (of the relevant vocabulary) such that if any two structures (of the relevant vocabulary) satisfy such a sentence, then they are isomorphic. We will conclude from this fact that isomorphism and elementary equivalence are equal (in finite vocabularies and finite structures), which we will use in our proof of Lindström's Theorem. Following that, we will see a generalisation of "isomorphism": "partial isomorphism", and we shall see a useful technique in Model Theory, called the Back-and-Forth method, which utilises partial isomorphisms, and is used to show that two countably infinite structures are isomorphic.

Theorem 5.2.1. *Let τ be a finite vocabulary. If $\mathfrak{M}$ is a τ-structure, and $Dom(\mathfrak{M})$ is finite, then there is a sentence $\phi \in \mathcal{L}_{\omega,\omega}(\tau)$ such that $\mathfrak{M} \models_{\mathcal{L}_{\omega,\omega}} \phi$ and for any τ-structure $\mathfrak{N}$,*

$$\textit{if} \quad \mathfrak{N} \models_{\mathcal{L}_{\omega,\omega}} \phi, \ \textit{then} \quad \mathfrak{N} \cong \mathfrak{M}.$$

Proof: denote the elements of $\mathrm{Dom}(\mathfrak{M})$ as $\{m_1, \dots, m_k\}$. Then, define

$$\begin{aligned}\phi := \exists x_1 \cdots \exists x_k (&\bigwedge_{i \neq j}^{k} x_i \neq x_j \wedge \\ &\forall x (\bigvee_{i}^{k} x = x_i) \wedge \\ &\bigwedge_{\substack{c \in \mathrm{Const}(\tau); \\ \iota_{\mathfrak{M}}(c) = m_i}} c = x_i \wedge \\ &\bigwedge_{\substack{R \in \mathrm{Rel}_l(\tau); \\ \langle m_{i_1}, \dots, m_{i_l} \rangle \in \iota_{\mathfrak{M}}(R)}} R x_{i_1} \cdots x_{i_l} \wedge \\ &\bigwedge_{\substack{R \in \mathrm{Rel}_l(\tau); \\ \langle m_{i_1}, \dots, m_{i_l} \rangle \notin \iota_{\mathfrak{M}}(R)}} \neg R x_{i_1} \cdots x_{i_l} \wedge \\ &\bigwedge_{\substack{f \in \mathrm{Func}_l(\tau); \\ \iota_{\mathfrak{M}}(f)(m_{i_1}, \dots, m_{i_l}) = x_j}} f(x_{i_1}, \dots, x_{i_l}) = x_j).\end{aligned}$$

This is clearly a finite sentence, as τ is finite, and so is the domain of $\mathfrak{M}$. Hence, $\phi \in \mathcal{L}_{\omega,\omega}(\tau)$.

Note that the first two lines encode the fact that the structure has precisely k elements, the third line encodes to which elements each constant symbol "picks out", the fourth line encodes for which elements of the domain each relation holds for, the fifth encodes for which elements of the domain each relation does not hold for, and the final line encodes to where each function maps, given any input-tuple of elements of the domain. Clearly, these are all facts about $\mathfrak{M}$, and so $\mathfrak{M} \vDash_{\mathcal{L}_{\omega,\omega}} \phi$.

Furthermore, we can see that if $\mathfrak{N} \vDash_{\mathcal{L}_{\omega,\omega}} \phi$, then by the first two lines, the domain of $\mathfrak{N}$ has the same cardinality as the domain of $\mathfrak{M}$. Then, if we remove the existential quantifiers of ϕ to obtain a formula ψ, with free variables $x_1, \dots, x_k$, and we order $\mathrm{Dom}(\mathfrak{N})$ so that $\mathrm{Dom}(\mathfrak{N}) = \{n_1, \dots, n_k\}$, and $\psi(n_1, \dots, n_k)$, then it is clear that $m_i \mapsto n_i$ is an isomorphism between the two structures. ■

From this, we can conclude:

Corollary 5.2.2. *Let τ be a finite vocabulary. If $\mathfrak{M}$ is a τ-structure, with Dom$(\mathfrak{M})$ finite, then for any τ-structure $\mathfrak{N}$, we have*

$$\mathfrak{M} \equiv \mathfrak{N} \quad \textit{if and only if} \quad \mathfrak{M} \cong \mathfrak{N}.$$

Proof: for the forward direction, let ϕ be as in Theorem 5.2.1, then $\mathfrak{M} \vDash_{\mathcal{L}_{\omega,\omega}} \phi$ if and only if $\mathfrak{N} \vDash_{\mathcal{L}_{\omega,\omega}} \phi$, but we know that $\mathfrak{M} \vDash_{\mathcal{L}_{\omega,\omega}} \phi$, and so $\mathfrak{N} \vDash_{\mathcal{L}_{\omega,\omega}} \phi$. Hence, $\mathfrak{M} \cong \mathfrak{N}$.

For the backward direction, note that if $\mathfrak{M} \cong \mathfrak{N}$, then $\mathfrak{M}$ and $\mathfrak{N}$ agree on all the atomic sentences, and so when inductively checking for the truth of a sentence of First-Order Logic, we know that all the atomic sentences will agree, and so any sentence will agree. ■

This tells us that for finite vocabularies and structures, isomorphisms and elementary equivalence are equivalent. So, to show one, we can just show the other.

This does not hold true, in general, for structures with infinite domains (it is difficult to give examples with the theory developed in this project).

5.2.1 The Back-and-Forth Method

Finally, we shall introduce the notion of partial isomorphisms and introduce the Back-and-Forth method, which is a method for demonstrating that there is an isomorphism between countably infinite structures. This is the final piece of theory we need before going on to prove Lindström's Theorem in the next section.

The proofs and definitions in this section are edited from [Ebb16, pp. 53–58]. But, Cantor's Theorem is expanded from [Hod97, p. 79].

Definition 5.2.3 (Partial Isomorphism). Let τ be a vocabulary, and $\mathfrak{M}$ and $\mathfrak{N}$ be τ-structures. Then, we say p is a *partial isomorphism* between $\mathfrak{M}$ and $\mathfrak{N}$ if and only if the following conditions are satisfied:

- p is a partial function[11] between $\mathrm{Dom}(\mathfrak{M})$ and $\mathrm{Dom}(\mathfrak{N})$;
- p is defined on only finitely many elements of $\mathrm{Dom}(\mathfrak{M})$ (including none);
- p is injective (that is, $p(x) = p(y) \Rightarrow x = y$);
- for each $c \in \mathrm{Const}(\tau)$, if $p(\iota_{\mathfrak{M}}(c))$ is defined, then $p(\iota_{\mathfrak{M}}(c)) = \iota_{\mathfrak{N}}(c)$;
- for each positive integer n, each $f \in \mathrm{Func}_n(\tau)$, and each n-tuple $(m_1, m_2, \dots, m_n)$ such that each element of the tuple has a defined mapping under p,

$$p(\iota_{\mathfrak{M}}(f)(m_1, m_2, \dots, m_n)) = \iota_{\mathfrak{N}}(f)(p(m_1), p(m_2), \dots, p(m_n));$$

  and

- for each positive integer n, each $R \in \mathrm{Rel}_n(\tau)$, and each n-tuple $(m_1, m_2, \dots, m_n)$ such that each element of the tuple has a defined mapping under p,

$$\iota_{\mathfrak{M}}(R)m_1m_2\cdots m_n \quad \text{if and only if} \quad \iota_{\mathfrak{N}}(R)p(m_1)p(m_2)\cdots p(m_n).$$

Informally, a partial isomorphism from a τ-structure $\mathfrak{M}$ to a τ-structure $\mathfrak{N}$ is an isomorphism (technically, a σ-isomorphism, where $\sigma \subseteq \tau$), which disregards constants that do not appear in its domain, from a subset of $\mathrm{Dom}(\mathfrak{M})$ onto its own image (a subset of $\mathrm{Dom}(\mathfrak{N})$).

It is useful to note that the empty function is a partial isomorphism between any two structures.

Definition 5.2.4 (Partially Isomorphic Structures). Given a vocabulary τ, we say that two τ-structures $\mathfrak{M}$ and $\mathfrak{N}$ are *partially isomorphic* if and only if there is a non-empty set I consisting of partial isomorphisms between $\mathfrak{M}$ and $\mathfrak{N}$, such that I satisfies both the *back property* and the *forth property*:

- (Forth property) For each $p \in I$ and $m \in \mathrm{Dom}(\mathfrak{M})$, there exists a $q \in I$ such that p is a "subfunction" of q[12], and q is defined on m.
- (Back property) For each $p \in I$ and $n \in \mathrm{Dom}(\mathfrak{N})$, there exists a $q \in I$ such that p is a "subfunction" of q, and there is some $m \in \mathrm{Dom}(\mathfrak{M})$ such that $q(m)$ is defined, and $q(m) = n$.

[11] A partial function is exactly like a function, except not every element of the domain has to have a defined image. And it will be useful to note that the empty function ($f : \emptyset \to \emptyset$) is always a partial function, even when we expand the domain and range.

[12] What we mean by this is that for all x such that $p(x)$ is defined, $q(x)$ is defined and $p(x) = q(x)$.

Given this definition, we can now show that any two countably-infinite partially isomorphic structures are isomorphic. In doing so, we will have demonstrated a technique for proving that two countably-infinite structure are isomorphic. This is the Back-and-Forth technique. We shall explain it, in full, and provide an example, after the next theorem.

Theorem 5.2.5. *Let τ be a vocabulary, and let $\mathfrak{M}$ and $\mathfrak{N}$ be partially isomorphic τ-structures with countably-infinite domains, then $\mathfrak{M} \cong \mathfrak{N}$.*

Proof: enumerate (without repeats) $\mathrm{Dom}(\mathfrak{M})$ as $m_0, m_1, \ldots$ and $\mathrm{Dom}(\mathfrak{N})$ as $n_0, n_1, \ldots$. Then, let $p_0 \in I$. We will define a sequence of partial isomorphisms $p_0, p_1, \ldots$ such that each p_i is a "subfunction" of p_{i+1} (and, consequently of each p_j for $j \geq i$), by the following:

- if $k + 1$ is odd (and $k = 2q$), then, by the forth property, we can find a p_{k+1} in I such that p_k is a "subfunction" of p_{k+1}, and p_{k+1} is defined on m_q; and
- if $k + 1$ is even (and $k + 1 = 2q$), then, by the back property, we can find a p_{k+1} in I such that p_k is a "subfunction" of p_{k+1}, and there is some $m \in \mathrm{Dom}(\mathfrak{M})$ such that $p_{k+1}(m)$ is defined, and $p_{k+1}(m) = n$.

We can use this sequence to induce a τ-isomorphism between $\mathfrak{M}$ and $\mathfrak{N}$ as follows: let $p : \mathrm{Dom}(\mathfrak{M}) \to \mathrm{Dom}(\mathfrak{N})$ be such that $m \mapsto p_i(m)$, where i is the least natural number such that $p_i(m)$ is defined in our sequence.

We will now show that this is indeed a τ-isomorphism.

- To see injectivity, note that if $p(x) = p(y)$, then we know that for some natural numbers i and j, $p_i(x) = p_j(y)$, and so, if, without loss of generality, i is the maximum of i and j, then we know that $p_i(x) = p_j(y) = p_i(y)$, as we have a sequence of "subfunctions", but, then as these are partial isomorphisms, p_i is injective, and so $x = y$.
- To see surjectivity, let $n_k \in \mathrm{Dom}(\mathfrak{N})$, then, by construction, there exists some $m \in \mathrm{Dom}(\mathfrak{M})$ such that $p_{2k}(m) = n_k$, and so we know that $p(m) = n_k$, because the p_is are a sequence of "subfunctions".
- To see that "constants map to the corresponding constants", note that at some point, we will have met the interpretation of each constant symbol, as p is bijective, by the previous two bullet points, and when we do, we know that it maps to the interpretation of the corresponding constant symbol, as each p_i is a partial isomorphism.
- A similar argument can be applied for relations and functions.

■

The Back-and-Forth method, then is to use Theorem 5.2.5, by showing that two countably-infinite structures are partially isomorphic.

For example, Cantor proved (although not using this method) that all countable dense linear orderings, without endpoints, are isomorphic. Note that all dense linear orderings are infinite, so countable dense linear orderings are countably-infinite, and also note that the rationals are a countable dense linear ordering, without endpoints, so this theorem is saying that every countable dense linear ordering is isomorphic to the rationals. We shall provide a proof using the Back-and-Forth method.

Theorem 5.2.6 (Cantor). *Let $\tau = \{<\}$, and let $\mathfrak{M}$ and $\mathfrak{N}$ be τ-structures, which are also countable dense linear orderings, without endpoints. Then, $\mathfrak{M} \cong \mathfrak{N}$.*

Proof: we use the Back-and-Forth method, by showing that $\mathfrak{M}$ and $\mathfrak{N}$ are partially isomorphic, and then we conclude the desired result by applying Theorem 5.2.5.

Let I be the set of partial isomorphisms between $\mathfrak{M}$ and $\mathfrak{N}$. We know that the empty function is in I and so it is non-empty.

We will show that I satisfies the forth property. So, let $p \in I$, and list the elements of $\mathrm{Dom}(\mathfrak{M})$ on which p is defined in a way such that $m_1 \; \iota_{\mathfrak{M}}(<) \; m_2 \; \iota_{\mathfrak{M}}(<) \; \cdots \; \iota_{\mathfrak{M}}(<) \; m_k$, where k is the total number of elements on which p is defined (which is finite, by definition). Then, let $m \in \mathrm{Dom}(\mathfrak{M})$, then choose $n \in \mathrm{Dom}(\mathfrak{N})$ such that:

- if $m \; \iota_{\mathfrak{M}}(<) \; m_1$, then $n \; \iota_{\mathfrak{N}}(<) \; p(m_1)$;
- if $m_k \; \iota_{\mathfrak{M}}(<) \; m$, then $p(m_k) \; \iota_{\mathfrak{N}}(<) \; n$; and
- if $m_i \; \iota_{\mathfrak{M}}(<) \; m \; \iota_{\mathfrak{M}}(<) \; m_{i+1}$, then $p(m_i) \; \iota_{\mathfrak{M}}(<) \; n \; \iota_{\mathfrak{M}}(<) \; p(m_{i+1})$.

We can justify that it is always possible to find such an n, because $\mathfrak{N}$ is a dense linear ordering, without endpoints. It follows that if we let p' be the same partial function as p, but also define p' to be defined on m so that $p'(m) = n$, then p' is also, obviously, a partial isomorphism, by our choice of n. Thus, we see that I has the forth property.

We can use a very similar argument to demonstrate the back property. □

6 Lindström's Theorem

We shall now see a proof of Lindström's theorem. This is an important result in Abstract Model Theory. The theorem gives a neat characterisation of the expressive power of First-Order Logic. It says that if any Logic is at least as expressive as First-Order Logic, and satisfies both ω-Compactness, and has the $\aleph_0$-Downward-Löwenheim-Skolem-Tarski Property, then that Logic must, in fact, be First-Order Logic. Hence, we know that these properties characterise First-Order Logic, in the sense that it is the most expressive Logic to have such properties.

Lindström's Theorem, then, is the foundational result in Abstract Model Theory, because it characterises, completely, First-Order Logic, which is our prime example of a Logic. So, as the field is about characterising and comparing Logics, Lindström's Theorem is a very nice result.

The proof proceeds in three steps. In the first step, we show that for ω-Compact Orthodox Logics, at least as strong as First-Order Logic, each sentence depends on at most finitely many symbols from the vocabulary.

Then, we show that if we were to extend First-Order Logic with a new, previously inexpressible sentence, and if the resulting Logic were still to have the Countable Compactness Property and the $\aleph_0$-Downward-Löwenheim-Skolem-Tarski Property (and still be Orthodox), then we would be able to find countably-infinite elementarily equivalent structures, one in which our new sentence holds, and one in which is doesn't, both with the same domain.

Finally, we will show that in addition to the properties from the previous paragraph, we can find structures which are isomorphic in the reduct of the finite fragment of our vocabulary that decides the truth of our new sentence (which we proved must exist, by our first step), demonstrating a contradiction.

Hence, we will conclude that First-Order Logic is the strongest Orthodox Logic such that it has the Countable Compactness Property, and the $\aleph_0$-Downward-Löwenheim-Skolem-Tarski Property. This is Lindström's Theorem.

The proofs in this section are based on those given in [Flu16, pp. 79–81], but with many details filled in.

6.1 The Proof

For this section, we let $\cdot^\star$ be a renaming (where the vocabulary is apparent from context), such that the domain and range are disjoint; we also fix $\cdot^\dagger$ to be the inverse of $\cdot^\star$.

We now demonstrate the first result, that in ω-Compact Orthodox Logics, at least as strong as First-Order Logic, any sentence depends on only a finite fragment of the vocabulary.

Lemma 6.1.1. *Let τ be a vocabulary and $\mathcal{L} \geq \mathcal{L}_{\omega,\omega}$ an Orthodox Logic with the Countable Compactness Property. Then, given $\psi \in \mathcal{L}(\tau)$, there is a finite vocabulary $\tau_0 \subseteq \tau$ such that for any τ-structures $\mathfrak{M}$ and $\mathfrak{N}$,*

$$\mathfrak{M} \upharpoonright \tau_0 \cong \mathfrak{N} \upharpoonright \tau_0 \Rightarrow (\mathfrak{M} \vDash_{\mathcal{L}} \psi \Leftrightarrow \mathfrak{N} \vDash_{\mathcal{L}} \psi).$$

Proof: let $\Phi \subseteq \mathcal{L}(\tau \cup \tau^\star)$ be

$$\begin{aligned}\Phi := &\{\forall x_1 \cdots \forall x_n (Rx_1 \cdots x_n \leftrightarrow R^\star x_1 \cdots x_n); n \geq 1, R \in \mathrm{Rel}_n(\tau)\}\\ &\cup \{\forall x_1 \cdots \forall x_n f(x_1, \dots, x_n) = f^\star(x_1, \dots, x_n); n \geq 1, f \in \mathrm{Func}_n(\tau)\}\\ &\cup \{c = c^\star; c \in \mathrm{Const}(\tau)\}.\end{aligned}$$

Note, then, that clearly $\Phi \vDash_{\mathcal{L}} \psi \leftrightarrow \psi^\star$ – as Φ specifies completely everything *about* a structure. Hence, by Countable Compactness, there is a finite set of sentences $X \subseteq \Phi$ such that

$$X \vDash_{\mathcal{L}} \psi \leftrightarrow \psi^\star.$$

Now, take σ as the, necessarily finite, set of symbols appearing in sentences in X (note that this is a subset of $\tau \cup \tau^\star$). Define $\sigma' := (\sigma \setminus \tau^\star) \cup (\sigma \setminus \tau)^\dagger$; so, $\sigma' \subseteq \tau$ and σ' is finite.

Then, suppose $\mathfrak{M}$ and $\mathfrak{N}$ are τ-structures such that $\mathfrak{M} \upharpoonright \sigma' \cong \mathfrak{N} \upharpoonright \sigma'$, and we may assume (by the isomorphism property, as we only care about truth in the models, so we are free to re-arrange things as we wish, as long as this is preserved) that $\mathfrak{M} \upharpoonright \sigma' = \mathfrak{N} \upharpoonright \sigma'$.

Now, we can see that $\mathfrak{M} \sqcup \mathfrak{N}^\star \vDash_{\mathcal{L}} X$, because $\mathfrak{M}$ and $\mathfrak{N}$ agree on all the σ'-sentences, and so every σ'-sentence holds in the joint structure if and only if the $(\sigma')^\star$-sentence holds – i.e., every sentence of X holds. Consequently, $\mathfrak{M} \sqcup \mathfrak{N}^\star \vDash_{\mathcal{L}} \psi \leftrightarrow \psi^\star$. So, we conclude that $\mathfrak{M} \vDash_{\mathcal{L}} \psi$ if and only if $\mathfrak{N}^\star \vDash_{\mathcal{L}} \psi^\star$; for, otherwise, if without loss of generality, $\mathfrak{M} \nvDash_{\mathcal{L}} \psi$ but $\mathfrak{N}^\star \vDash_{\mathcal{L}} \psi^\star$, then $\mathfrak{M} \sqcup \mathfrak{N}^\star \nvDash_{\mathcal{L}} \psi \leftrightarrow \psi^\star$ as $\mathcal{L}$ is Orthodox, which would contradict the previous sentence.

Similarly, by the renaming property, we have $\mathfrak{N}^\star \vDash_{\mathcal{L}} \psi^\star$ if and only if $\mathfrak{N} \vDash_{\mathcal{L}} \psi$. Hence, $\mathfrak{M} \vDash_{\mathcal{L}} \psi$ if and only if $\mathfrak{N} \vDash_{\mathcal{L}} \psi$. Thus, σ' is such a finite vocabulary satisfying the conditions of the lemma. ■

We will now use Lemma 6.1.1 to show that if an Orthodox Logic is stronger than First-Order Logic, and has both the Countable Compactness, and the $\aleph_0$-Downward-Löwenheim-Skolem-Tarski Properties, then we can find countably-infinite structures which agree on all the First-Order sentences, but disagree on a given sentence that is not equivalent to any First-Order sentence.

Lemma 6.1.2. *Let τ be a vocabulary, and let $\mathcal{L} \geq \mathcal{L}_{\omega,\omega}$ be an Orthodox Logic with both the $\aleph_0$-Downward-Löwenheim-Skolem-Tarski Property and the Countable Compactness Property. Then, let $\psi \in \mathcal{L}(\tau)$ be not equivalent to any first-order sentence. Then there exist elementarily equivalent countably-infinite structures $\mathfrak{M}$ and $\mathfrak{N}$ (on the same domain) such that*

$$\mathfrak{M} \vDash_{\mathcal{L}} \psi \quad \text{and} \quad \mathfrak{N} \vDash_{\mathcal{L}} \neg\psi.$$

Proof: let τ be a vocabulary, and $\psi \in \mathcal{L}(\tau)$ be not equivalent to any first-order sentence. Then, choose a finite vocabulary $\tau_0 \subseteq \tau$ such that it satisfies Lemma 6.1.1 with our chosen ψ. Then, enumerate $\mathcal{L}_{\omega,\omega}(\tau_0)$ as $\phi_1, \phi_2, \ldots$. By induction, using Theorem 5.1.6, which says that if ψ is not equivalent to any first-order sentence, then neither $\psi \wedge \phi$ nor $\psi \wedge \neg\phi$ is, for any first-order sentence ϕ, because $\mathcal{L}$ is Orthodox, we conclude that there is an enumeration $\psi_1, \psi_2, \ldots$ of sentences such that each $\psi_i \in \{\phi_i, \neg\phi_i\}$, and $\psi \wedge \psi_1 \wedge \cdots \wedge \psi_n$ is not equivalent to a first-order sentence, for any $n \in \mathbb{N}$. Similarly, $\neg\psi \wedge \psi_1 \wedge \cdots \wedge \psi_n$ is not equivalent to a first-order sentence either (we proved this fact in Theorem 5.1.7; again, as $\mathcal{L}$ is Orthodox). And, so, by Theorem 5.1.5, which says that if a sentence if not equivalent to a First-Order sentence it is satisfiable, both are satisfiable.

Define $\Psi := \{\psi_n;\ n \in \mathbb{Z}^+\}$. Then, by ω-Compactness (and based on the fact that $\mathcal{L}$ is Orthodox, and the properties of Orthodox conjunction), there exist τ-structures $\mathfrak{M}$ and $\mathfrak{N}$, such that

$$\mathfrak{M} \vDash_{\mathcal{L}} \Psi \cup \{\psi\},$$

and

$$\mathfrak{N} \vDash_{\mathcal{L}} \Psi \cup \{\neg\psi\},$$

(for each finite subset X of Ψ, there is a maximum $k \in \omega$ such that $\psi_k \in X$; then, any model of $\psi \wedge \psi_1 \wedge \cdots \wedge \psi_k$ is a model of X, by our notational definition of conjunction, of which there must be at least one, as this conjunctive sentence is satisfiable; a similar argument holds for $\Psi \cup \{\neg\psi\}$) and by the $\aleph_0$-Downward-Löwenheim-Skolem-Tarski Property, we may assume that $\mathfrak{M}$ and $\mathfrak{N}$ are countably-infinite structures.

But, we know that $\mathfrak{M} \restriction \tau_0 \equiv \mathfrak{N} \restriction \tau_0$, by construction, and, by Lemma 6.1.1, we must have that $\mathfrak{M} \restriction \tau_0 \not\cong \mathfrak{N} \restriction \tau_0$. Also, by construction, we have that $\mathfrak{M} \vDash \psi$ and $\mathfrak{N} \vDash \neg\psi$. Hence, by Corollary 5.2.2, which says that for finite structures, isomorphism and elementary equivalence coincide, we must have that $|\mathfrak{M}| = |\mathfrak{N}| = \aleph_0$. And, without loss of generality, we may assume that $\mathrm{dom}(\mathfrak{M}) = \mathrm{dom}(\mathfrak{N})$. ■

Finally, we shall show that we can find structures, with the same properties as in Lemma 6.1.2, but also have isomorphic τ_0-reducts, which we will conclude is a contradiction.

Lemma 6.1.3. *Let τ be a vocabulary, and let $\mathcal{L} \geq \mathcal{L}_{\omega,\omega}$ be an Orthodox Logic with both the $\aleph_0$-Downward-Löwenheim-Skolem-Tarski Property and the Countable Compactness Property. Then, let*

$\psi \in \mathcal{L}(\tau)$ be not equivalent to any first-order sentence. If $\tau_0 \subseteq \tau$ satisfies Lemma 6.1.1 (with ψ), then, there exist elementarily equivalent isomorphic structures $\mathfrak{M} \upharpoonright \tau_0$ and $\mathfrak{N} \upharpoonright \tau_0$ with

$$\mathfrak{M} \upharpoonright \tau_0 \vDash_{\mathcal{L}} \psi \text{ and } \mathfrak{N} \upharpoonright \tau_0 \vDash_{\mathcal{L}} \neg\psi.$$

Proof: first, choose a finite vocabulary $\tau_0 \subseteq \tau$ such that it satisfies Lemma 6.1.1, with our chosen ψ, like in Lemma 6.1.2. Then, choose, for each $n \in \mathbb{N}$, new $(2n+1)$-ary function symbols f_n and g_n.

Set $\tau' := \tau \cup \tau^\star \cup \{f_n;\ n \in \mathbb{N}\} \cup \{g_n;\ n \in \mathbb{N}\}$. Note that τ' is countable as it is a countable union of countable sets.

Now, we set X to be the (countable) set of the following $\mathcal{L}(\tau')$-sentences:

- ψ (which expresses the fact that the τ-reduct is a model of ψ, because ψ is a $\mathcal{L}(\tau)$-sentence);
- $\neg\psi^\star$ (which expresses the fact that the $\tau^\star$-reduct is a model of $\neg\psi^\star$); and
- for each $\mathcal{L}_{\omega,\omega}(\tau_0)$-sentence ϕ, the sentence $\phi \leftrightarrow \phi\star$ (which expresses the fact that the τ_0-reduct and the $\tau_0^\star$-reduct are elementarily equivalent).

Note that every finite subset of X is satisfiable, because we have just seen, in Lemma 6.1.2 a very similar situation: two elementarily equivalent models, one modelling ψ and one modelling $\neg\psi$, so taking $\mathfrak{M}$ and $\mathfrak{N}$ from that Lemma, we see that $\mathfrak{M} \sqcup \mathfrak{N}^\star$ models X (and we note that our resulting structure is countably-infinite).

We shall now see that we can add sentences to X that also enforce that the τ_0 and $\tau_0^\star$ reducts are isomorphic.

Next, for each $n \in \mathbb{N}$, select an enumeration of all the (logically distinct) $\mathcal{L}_{\omega,\omega}(\tau_0)$-formulæ with at most $n+1$ free variables, of which there are finitely many[13] (denoted κ_n): $\eta_1^n, \eta_2^n, \dots, \eta_{\kappa_n}^n$. Then, consider the following (countable) set Γ of $\mathcal{L}(\tau')$-sentences:

- for each $n \in \mathbb{N}$,

$$\forall x_1 \cdots \forall x_n \forall y_1 \cdots \forall y_n \forall x \left(\exists y \left(\bigwedge_{i=0}^{\kappa_n} (\eta_i^n(x_1, \dots, x_n, x) \leftrightarrow \eta_i^{n\star}(y_1, \dots, y_n, y)) \right) \right.$$
$$\left. \to \bigwedge_{i=0}^{\kappa_n} (\eta_i^n(x_1, \dots, x_n, x) \leftrightarrow \eta_i^{n\star}(y_1, \dots, y_n, f_n(x_1, \dots, x_n, y_1, \dots, y_n, x))) \right),$$

  which essentially says that if there is a y which satisfies the same $\mathcal{L}_{\omega,\omega}(\tau_0)$-formulæ (with at most $n+1$ free variables) as x (except starred), when provided with other variables, then f_n maps to such a y, when it is given the same variables, and given x; and

- for each $n \in \mathbb{N}$,

$$\forall x_1 \cdots \forall x_n \forall y_1 \cdots \forall y_n \forall x \left(\exists y \left(\bigwedge_{i=0}^{\kappa_n} (\eta_i^n(x_1, \dots, x_n, x) \leftrightarrow \eta_i^{n\star}(y_1, \dots, y_n, y)) \right) \right.$$
$$\left. \to \bigwedge_{i=0}^{\kappa_n} (\eta_i^n(x_1, \dots, x_n, g_n(x_1, \dots, x_n, y_1, \dots, y_n, y)) \leftrightarrow \eta_i^{n\star}(y_1, \dots, y_n, y)) \right),$$

  which is obviously similar to the previous, except g_n gives us such an x when provided with the other variables and y.

[13]See, for example, PY4612 Advanced Logic.

Then, we note that given a finite set of sentences from Γ, we can expand an arbitrary $(\tau_0 \cup \tau_0^\star)$-structure into a model of Γ, because in satisfying the sentences in Γ, all we care about is where f_n and g_n map to, which does not interfere with any sentences not containing such function symbols (which no sentence of a $(\tau \cup \tau^\star)$-structure can). Therefore, every finite subset of $X \cup \Gamma$ (a countable set) is satisfiable. So, by ω-Compactness, $X \cup \Gamma$ is satisfiable, and by the $\aleph_0$-Downward-Löwenheim-Skolem-Tarski, there is a countably-infinite model (see Lemma 6.1.2), $\mathfrak{D}$ of $X \cup \Gamma$.

If we let $\mathfrak{M} := \mathfrak{D} \restriction \tau$ and $\mathfrak{N} := (\mathfrak{D} \restriction \tau^\star)^\dagger$, then $\mathrm{dom}(\mathfrak{D}) = \mathrm{dom}(\mathfrak{M}) = \mathrm{dom}(\mathfrak{N})$, $\mathfrak{M} \vDash_{\mathcal{L}} \psi$, $\mathfrak{N} \vDash_{\mathcal{L}} \neg\psi$, and $\mathfrak{M} \restriction \tau_0 \equiv \mathfrak{N} \restriction \tau_0$, by our sentences in X. And, furthermore, we shall show that $\mathfrak{M} \restriction \tau_0 \cong \mathfrak{N} \restriction \tau_0$, using the sentences in Γ by the back-and-forth method. Note, that both $\mathfrak{M} \restriction \tau_0$ and $\mathfrak{N} \restriction \tau_0$ are countably-infinite as they share their domains with $\mathfrak{D}$.

Enumerate, without repeats, $\mathrm{dom}(\mathfrak{D})$ as $d_1, d_2, \ldots$. Then, because $\mathfrak{M} \restriction \tau_0 \equiv \mathfrak{N} \restriction \tau_0$, it follows, from our sentences in Γ, that we can construct the following sequence of facts about elementary equivalence:

$$\begin{aligned}
\mathfrak{M} \restriction \tau_0 \sqcup \langle \mathrm{dom}(\mathfrak{D}); d_1 \rangle &\equiv \mathfrak{M} \restriction \tau_0 \sqcup \langle \mathrm{dom}(\mathfrak{D}); f_0(d_1) \rangle \\
\mathfrak{M} \restriction \tau_0 \sqcup \langle \mathrm{dom}(\mathfrak{D}); d_1, g_1(d_1, f_0(d_1), d_1) \rangle &\equiv \mathfrak{M} \restriction \tau_0 \sqcup \langle \mathrm{dom}(\mathfrak{D}); f_0(d_1), d_1 \rangle \\
\mathfrak{M} \restriction \tau_0 \sqcup \langle \mathrm{dom}(\mathfrak{D}); d_1, g_1(d_1, f_0(d_1), d_1), d_2 \rangle &\equiv \mathfrak{M} \restriction \tau_0 \sqcup \langle \mathrm{dom}(\mathfrak{D}); f_0(d_1), d_1, f_1(d_1, f_0(d_1), d_2) \rangle \\
&\vdots
\end{aligned}$$

that is, as we gradually add each element of the domain as a constant (to either of the structures), we know how to choose an element of the other structure such that both elements of the domain satisfies the same $\mathcal{L}_{\omega,\omega}(\tau_0)$-sentences. Therefore, we can use this mapping to construct a sequence of partial isomorphisms, which obviously have the back property and the forth property. So, by definition, $\mathfrak{M} \restriction \tau_0$ and $\mathfrak{N} \restriction \tau_0$ are partially isomorphic. Thus, by Theorem 5.2.5, which says that countably-infinite partially isomorphic structures are isomorphic, $\mathfrak{M} \restriction \tau_0$ is isomorphic to $\mathfrak{N} \restriction \tau_0$. ∎

Finally, we can make clear the contradiction, and prove Lindström's Theorem:

Theorem 6.1.4 (Lindström's Theorem). *Let $\mathcal{L} \leq \mathcal{L}_{\omega,\omega}$ be an Orthodox Logic with the Countable Compactness and $\aleph_0$-Downward-Löwenheim-Skolem-Tarski Properties, then $\mathcal{L}$ is equivalent to $\mathcal{L}_{\omega,\omega}$.*

Proof: suppose otherwise, then there is some vocabulary τ such that there is a $\mathcal{L}(\tau)$-sentence ψ, which is not equivalent to any first-order sentence. But then, by Lemma 6.1.3, we can find two countably-infinite, isomorphic structures $\mathfrak{M}$ and $\mathfrak{N}$ such that $\mathfrak{M} \vDash_{\mathcal{L}} \psi$ and $\mathfrak{N} \vDash_{\mathcal{L}} \neg\psi$, which are both τ_0-structures, where τ_0 is a finite subset of τ, and, by Lemma 6.1.1, is such that if $\mathfrak{M} \equiv \mathfrak{N}$, then $\mathfrak{M} \vDash_{\mathcal{L}} \psi$ if and only if $\mathfrak{N} \vDash_{\mathcal{L}} \psi$. This is a contradiction, as we know that (due to the notational definition of negation for Orthodox Logics) that $\mathfrak{N} \nvDash_{\mathcal{L}} \psi$, but $\mathfrak{M} \vDash_{\mathcal{L}} \psi$, by construction. Hence, there cannot be such a sentence.

So, any such Orthodox Logic, which is stronger than First-Order Logic, must either violate Countable Compactness, or the $\aleph_0$-Downward-Löwenheim-Skolem-Tarski theorem. ∎

As we can see, then, Lindström's Theorem nicely characterises First-Order Logic in relation to all other Orthodox Logics (which are usually the main target of Abstract Model Theory).

6.2 Lindström Theorems and Abstract Model Theory

This concludes our introduction to Abstract Model Theory; but, in the spirit of classifying and characterising Logics (the goals of Abstract Model Theory), there are a host of other theorems that neatly characterise Logics. They are named, after Lindström's Theorem, "Lindström Theorems". There are other Lindström Theorems for First-Order Logic, which can also be seen in [Flu16, p. 82], characterising First-Order Logic with properties that have not been seen in this project.

Moving away from First-Order Logic, there are two other interesting classes of Logic that I suggest an interested reader look into: Modal and Intuitionistic Logics. For an introduction to Modal Logic (which I recommend an interested reader to look at first), I recommend the book [BRV01]. Then, for an introduction to Intuitionistic Logic (including an introduction to Heyting Algebras, which are used to define Intuitionistic Logic, and are a generalisation of Boolean Algebras), the lecture notes [BJ05] are nice. Then, if an interested reader wants to look at the Lindström Theorems for these Logics, a proof and statement of the Modal Lindström Theorem can be found in [Ben07]; and a proof and statement of the Intuitionistic Lindström Theorem can be found in [OBZ21].

Had there been more space in this project, I would have liked to provide an introduction to Modal and Intuitionistic Logics, and provided the aforementioned Lindström Theorems for the particular varieties described by the Lindström Theorems. Moreover, I wished that I could have included the deep relationship between Logics and games: we can express "truth in a structure", "satisfiability", and "Logical equivalence of structures" with deeply connected games; that is, we can play a games on structures to check if, for example in the first case, whether a given sentence holds in the structure. And, in doing so, we can characterise a Logic by its "truth games". Unfortunately, providing an introduction to games took us too far off of the main path, and seemed to confuse the main message, and so had to be cut. I strongly recommend that an interested reader explore this relationship, for example in a text like [Vää11].

Finally, for more pure Abstract Model Theory, then there is [BF13], which is a very big book, all about Abstract Model Theory. It is from this book that I have modelled our proof of Lindström's Theorem on, as well as our versions of Compactness, Löwenheim-Skolem-Tarski, Logic, and Orthodox Logic. Having read this project, an interested reader should be able to pick up and read this book.

References

[Bar16] J. Barwise. 'Chapter I: Model-Theoretic Logics: Background and Aims'. In: *Model-Theoretic Logics*. Perspectives in Logic. Cambridge University Press, Cambridge, 2016, pp. 3–23.

[Ben07] J. van Benthem. 'A New Modal Lindström Theorem'. In: *Logica Universalis* 1 (2007), pp. 125–138.

[BF13] J. Barwise, and S. Feferman. *Model-Theoretic Logics*. Perspectives in Logic. Cambridge University Press, Cambridge, 2013.

[BJ05] N. Bezhanishvili, and D. de Jongh. *Intuitionistic Logic*. 2005. url: http://www.cs.le.ac.uk/people/nb118/Publications/ESSLLI'05.pdf.

[BRV01] P. Blackburn, M. de Rijke, and Y. Venema. *Modal Logic*. Cambridge Tracts in Theoretical Computer Science. Cambridge University Press, Cambridge, 2001.

[Ebb16] H.-D. Ebbinghaus. 'Chapter II: Extended Logics: The General Framework'. In: *Model-Theoretic Logics*. Perspectives in Logic. Cambridge University Press, Cambridge, 2016, pp. 25–76.

[Flu16] J. Flum. 'Chapter III: Characterizing Logics'. In: *Model-Theoretic Logics*. Perspectives in Logic. Cambridge University Press, Cambridge, 2016, pp. 77–120.

[GH09] S. Givant, and P. Halmos. *Introduction to Boolean Algebras*. First. Undergraduate Texts in Mathematics. Springer, New York, 2009. isbn: 387402932.

[Hod97] W. Hodges. *A Shorter Model Theory*. Cambridge University Press, USA, 1997.

[Jec02] T. Jech. *Set Theory*. Third. Springer, Dordrecht, 2002.

[Kun13] K. Kunen. *Set Theory*. Revised edn. #34 of Studies in Logic. College Publications, London, 2013.

[Mar02] D. Marker. *Model Theory: An Introduction*. First. #217 of Graduate Texts in Mathematics. Springer, New York, 2002.

[Mar16] D. Marker. *Lectures on Infinitary Model Theory*. Lecture Notes in Logic. Cambridge University Press, Cambridge, 2016.

[OBZ21] G. Olkhovikov, G. Badia, and R. Zoghifard. *A Lindström Theorem for Intuitionistic First-Order Logic*. 2021. url: https://arxiv.org/abs/2103.17024.

[Vää11] J. Väänänen. *Models and Games*. #132 of Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 2011.