

Naïve Set/Class Theory and Second-Order Paraconsistent Logic

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1 Introduction

Naïve Set Theory is a theory of sets using at least the Axiom of Extensionality:

$$\forall x \forall y [x = y \leftrightarrow \forall z (z \in x \leftrightarrow z \in y)],$$

and the Axiom Schema of Naïve Comprehension, where $\phi(x)$ is a formula with all free variables among x :

$$\exists y \forall x [x \in y \leftrightarrow \phi(x)].$$

The former axiom asserts the main feature of sets: they are equal if and only if they have the same elements. I.e., sets are defined totally by their elements. The latter axiom schema asserts that for any formula with at most x free, there is a set corresponding to such a formula, in the sense that the set contains only and all those objects satisfying the formula.

The interest in Naïve Set Theory stems from the view that, as Weber [Web12, p. 288] puts it, “the naïve view of sets is as predicates in extension” (diacritics my own). However, classically, such a view is impossible. The typical example is the Russell Class – first presented in a letter to Frege, by Russell, demonstrating the issue with his Basic Law V, which is roughly equivalent to our modern Axiom Schema of Naïve Comprehension. The Russell Class is defined as

$$R := \{x; x \notin x\},$$

where this notation says that R is the unique set satisfying the Axiom Schema of Comprehension where $\phi(x) := x \notin x$. However, if such a class were to exist, then Naïve Set Theory, formulated classically, would be trivial, as the Russell Class can be used to create, classically, a contradiction: we see $R \in R$ implies $R \notin R$; and, we see $R \notin R$ implies $R \in R$. Both follow from the fact that R contains only and all those sets that do not contain themselves. From this, we can deduce that $R \in R$ if and only if $R \notin R$, and, due to the Law of Excluded Middle, we must have a contradiction.

So, supporters of Naïve Set Theory have been forced out of using classical logic. Perhaps the most naïve approach is of those who wish to simply accept that sets like the Russell Class exist, and, moreover, are contradictory. That is the tradition of using paraconsistent logics (i.e., a logic that has non-trivial models with true contradictions – i.e., denies the explosion or *ex falso quodlibet*) to formulate Naïve Set Theory. Notably, such philosophers include, Brady [Bra89], Priest [Pri06, Chapter 18], Restall [Res92], and Weber [Web10].

In this essay, I begin, in section 2, by saying that I believe, in addition to the Axiom of Extensionality, and the Axiom Schema of Naïve Comprehension, that models of Naïve Set Theory should also satisfy a principle that I call the “Principle of Definability”, which says that for each set there is a formula with at most x free such that the elements of the set are all and only those objects satisfying the formula. I call this “Candid Set Theory”.

From there on, I outline why second-order logic may be the best place to turn to construct such models. And, starting from section 3, I outline a basic second-order paraconsistent logic, which we modify from there on to produce a (hopefully) more suitable logic for the task. Notably, I will not attempt to construct a model of Naïve Set Theory in this logic, let alone Candid Set Theory. As we shall see, in section 4 and section 5, there are problems with second-order paraconsistent logics that first need to be settled before they can be used for such a task.

In section 4, I show that it is unfeasible to quantify over all possible paraconsistent predicates, and instead suggest that we quantify over only classical predicates and the predicates in the language.

Finally, in section 5, I explore an alternative method of validating the Axiom Schema of Naïve Comprehension, given we can no longer quantify over all formulæ of the logic, due to our results in section 4, as is often possible with second-order logics. This gives a more classical flavour to Naïve Set Theory, which may be regarded as a disadvantage to some. This also raises questions about our semantics, which I use to justify a final version of our basic logic for (hopefully) future use in the field of Candid Set Theory formulated using second-order paraconsistent logics.

1.1 Notation

Throughout, I shall use the $\wedge$ for conjunction, $\vee$ for disjunction, $\neg$ for negation, $\forall$ for universal quantification, and $\exists$ for existential quantification. First-Order variables will be written as lowercase latin letters, usually from the end of the alphabet: $x, y, z, \mathcal{E}c$, and second-order variable will be written as uppercase latin letters, again, usually from the end of the alphabet: $X, Y, Z, \mathcal{E}c$. Often, the arity of the predicate is written as a super script: for example $X^1, X^3, Y^{10}, \mathcal{E}c$. I will write $\phi(x)$ to mean that the free first-order (and free second-order) variables are at most the first-order variable x , and, similarly, I will write $\phi(X)$ to mean that the free second-order (and free first-order) variable are at most the second-order variable X .

2 Candid Set Theory

2.1 What?

I now argue for what I call “Candid Set Theory”¹. This extends Naïve Set Theory with a new principle, which I shall call the “Principle of Definability”:

Given a set S , there is a formula $\phi(x)$, of the underlying logic, where at most x is free, such that for any object y of the domain, $y \in S$ if and only if $\phi(y)$ holds.

Unless we are able to quantify over formulæ (or at least those formulæ with at most x free) in our logic, there is no way to add this as an axiom to Naïve Set Theory. I am not aware of any such systems. This seems like a *property of models*. In essence, the Principle of Definability says, to use typical model-theoretic terminology, every set is pointwise-definable in the model. Candid Set Theory, then, is a restriction on the models of Naïve Set Theory to only those models in which every set is pointwise-definable. Such models of ZF have been considered by Hamkins, Linetsky, and Reitz [HLR13]. Notably, they are necessarily countable models.

2.2 Why?

Why, then, do I think we ought to restrict our attention to such models, by adding the Principle of Definability to Naïve Set Theory? I will present three arguments; all rely on the fact the reader is already sensitive to the Naïve View. The first shall be directed to those who believe logic is a model of language – i.e., logical anti-realists. The second shall be directed to those who believe logic is a model of “consequence in nature” – i.e., logical realists. The third shall be more general.

The first argument, then, directed toward logical anti-realists who already accept Naïve Set Theory, is: it would seem bizarre if in our models of set theory, we had undefinable sets: for there would be no way to make inferences about the properties of an individual such set – only them all as a whole, through quantification. But this universe of sets is supposed to arise from logic as a model of language

¹Naïve Set Theory ain’t naïve ’nuff.

– and the naïve view of these sets is that they are, again, to quote Weber, “predicates in extension”. So, there would be sets in our models, which are supposedly “predicates in extension” but there would be no such predicate defining them. This seems ontologically unnecessary, and very bizarre. But, of course, this is not a water-tight argument and so I’m sure there are those who’d disagree.

The second argument is: if logic is a model of real, actual consequence in the world, then such a logic would not be able to make inferences about the individual sets, for there would be no way to pick out the individual (in fact, this view – the one of the second argument – seems to be making the claim that any model of the universe should be pointwise-definable, which is of course a very strong statement), and derive its properties. If the logic were able to do such a thing, then it must be able to pick out each set, using logical formulæ, which would make the set definable. This seems strange: for there are uncountably many real numbers, and I am claiming there are only countably many. We return to this reasoning shortly: I shall address the objection that Candid Set Theory restricts our attention to countable models.

Moreover, another objection to this second argument is that we only have epistemic access to a portion of true logical consequence: we cannot see the whole thing, and cannot appreciate the full power of logical consequence. Thus, those sets that were not pointwise-definable, may only be not pointwise-definable in virtue of the fact that we cannot reason about them, due to our limited epistemic access, but that the true logical consequence² of nature can, in fact, reason about such sets, and provide them with names.

The third is: surely “being a member of the set $\mathcal{S}$ ” is a predicate, and so this is the defining predicate of the set. However, notably, in writing this argument, I am giving $\mathcal{S}$ a name, which is assuming it is pointwise-definable. And so, taking what I’ve said at face-value can be seen to be question-begging. On the other hand, it seems like there is a naïve simplicity to this view, which is very appealing, at least to me: for every set, there is a predicate, which is “being a member of that set” – but, perhaps, this is assuming a name for the set, even in this case, in which case I am question-begging. I do not believe so, for I think such a predicate is mind and language independent (I am a logical realist), and so there is a real property (and hence predicate) of being a member of an arbitrary set, without requiring a name for the set. This is the most appealing, to myself, of the arguments I have given, but certainly may not convince anyone else!

Again, the objection that we only have access to a fragment of true logical consequence stands against this argument.

Another objection one might have to the view that Candid Set Theory should replace Naïve Set Theory is that I am essentially saying we want to restrict our attention to countable models of Naïve Set Theory – just as in the case with [HLR13]³. However, Read in conversation with Weber [Web12, p. 288] believes this is a good thing, as being uncountable is a limitative result. I however, would say the discussions about cardinality are not helpful: these are results about cardinality in a classical meta-theory. I believe that we want to use Candid Set Theory as our meta-theory, and re-formulate Candid Set Theory inside itself. Priest [Pri06, p. 259] agrees with this sentiment: “in the model theory of paraconsistent logic, we must therefore use paraconsistent set theory” – only I take the true “paraconsistent set theory” to be Candid Set Theory; and, as we shall see shortly, I propose model-theoretic semantics for our logic of choice.

In this case, we should only be worried about cardinality from the perspective of the logic itself –

²Or, indeed, logical consequences...

³There are countably many formulæ, so for there to exist a formula defining each set, there must be countably many sets.

and we know from our experience of Skolem’s paradox, which says there are countable models of set theory, and in those models, there are still “uncountable” sets – in the sense that there is no injection from such a set into the natural numbers *inside the model*. So, if such a countable model of ZF were our meta-theory, it does not seem to matter what a more general classical meta-theory would have to say about cardinality – only the “true meta-theory”, which I would take to be Candid Set Theory.

I think, then, the charge against Candid Set Theory that it only admits countable models is question begging: for someone would only accept this charge if they already accept a meta-theory different from Candid Set Theory. Of course, this response does not provide evidence for Candid Set Theory, it merely responds to the objection that the models are countable.

Even if one is not receptive to my arguments for restricting our attention totally to Candid Set Theory, Candid Set Theory can still be of interest to Naïve Set Theorists. This is in the same way that Hamkins, Linetsky, and Reitz were interested in looking at pointwise-definable models of ZF. We are interested in models where we can “talk” about everything that exists individually. Certainly, though, this does not provide philosophical justification for Candid Set Theory, but motivates it from a position of interest.

2.3 How?

How, if one is interested in the project of Candid Set Theory, should one go about it? It is not obvious that the Principle of Definability holds in the models of Naïve Set Theory already considered. E.g., the non-trivial models of Naïve Set Theory in the relevant logics DKQ and TKQ, constructed by Brady in [Bra06, p. 242]. And, if it’s anything like the issues with ZF discussed in [HLR13], then significant study is required of the current models of Naïve Set Theory, to check if the Principle of Definability holds.

So, I suggest in order to make it more likely that the Principle of Definability holds, we should take a look at second-order logic. Just as classical second-order logic dramatically ramps up the expressive power of classical first-order logic, the hope is that second-order paraconsistent logic will do the same for us. The hope, then, is that our models of Naïve Set Theory, will turn out to be the models of Candid Set Theory also. However, I do not have time to examine if we do get these results. That is something for future research. For now, this is a motivation for looking at Naïve Set Theory in a second-order paraconsistent context.

So, let us proceed, and see a basic paraconsistent logic, which we shall slowly make changes to.

3 A Basic Logic

I will now introduce a basic logic that we shall modify from here on. This is a logic which extends the quantified version of the Logic of Paradox (LP), which was first explored by Priest in [Pri79]. The semantics for this logic are based on those given by Priest in [Pri06, p. 85]. It would be ideal to be able to use LP as our logic of choice, extending it into the Second-Order, but, it has been shown by Forster [BFS13] (the paper also includes a strengthening of this result by Seligman, and some further thoughts from Beall) that LP cannot have any real conditional: if $\rightarrow$ is a conditional operator, and $A \rightarrow B, A \models B$, then, in fact, $A \rightarrow B \models B$. Which, of course, makes any conditional of LP useless for axiomatising, in any serious way, Naïve Set Theory. Hence, we proceed by adding our own conditional to LP.

3.1 First-Order Semantics

Our first parse of our logic will be the first-order fragment. Here, we are simply extending quantified LP (which is written as LPQ for short) with a suitable conditional. To do this, we use modal-semantics. The resulting logic will be called $\text{LPQ} \Rightarrow$ as we add the conditional ‘ $\Rightarrow$ ’ to LPQ.

First, we’ll define our well-formed formulæ. Given a vocabulary⁴ τ , we define the set of $\text{LPQ} \Rightarrow$ -formulæ given τ , denoted $\text{LPQ} \Rightarrow(\tau)$. We do not allow 0-ary functions or relations.

The terms of $\text{LPQ} \Rightarrow(\tau)$ are the smallest set satisfying:

- first-order variables: $x, y, z, \mathcal{E}c$. are terms of $\text{LPQ} \Rightarrow(\tau)$;
- if c is a constant symbol in τ , then c is a term of $\text{LPQ} \Rightarrow(\tau)$; and
- if f is an n -ary function symbol in τ , and $t_1, \dots, t_n$ are terms of $\text{LPQ} \Rightarrow(\tau)$, then $f(t_1, \dots, t_n)$ is a term of $\text{LPQ} \Rightarrow(\tau)$.

The atomic formulæ of $\text{LPQ} \Rightarrow(\tau)$ are the smallest set satisfying:

- if R is an n -ary relation symbol in τ , and $t_1, \dots, t_n$ are terms of $\text{LPQ} \Rightarrow(\tau)$, then $Rt_1 \dots t_n$ is an atomic formula of $\text{LPQ} \Rightarrow(\tau)$.

Then, $\text{LPQ} \Rightarrow(\tau)$ is the smallest set satisfying:

- the atomic formulæ of $\text{LPQ} \Rightarrow(\tau)$ are all in $\text{LPQ} \Rightarrow(\tau)$; and
- if $\phi, \psi \in \text{LPQ} \Rightarrow(\tau)$ and x is a variable, then $(\phi \wedge \psi)$, $\neg\phi$, $(\phi \Rightarrow \psi)$, and $\forall x\phi$ are in $\text{LPQ} \Rightarrow(\tau)$.

As usual, we omit brackets (or, use square brackets) when the resulting meaning is still clear. We treat $\phi \vee \psi$ as an abbreviation for $\neg(\neg\phi \wedge \neg\psi)$, and $\exists x\phi$ as an abbreviation for $\neg\forall x\neg\phi$. We say a variable is bound if it satisfies the standard conditions, and free otherwise; we call a formula with no free variable a sentence.

Given a vocabulary τ , we define a τ -structure of $\text{LPQ} \Rightarrow$ as a $\langle W, D, R, I, @ \rangle$ quintuple, consisting of W a set of worlds (which are arbitrary objects), D a non-empty set of objects, which shall act as our domain, R a binary relation on W , I an interpretation function that takes a world and an atomic sentence from $\text{LPQ} \Rightarrow(\tau^D)$ ⁵ and maps to to a non-empty subset of $\{0, 1\}$ – as well as interpreting, world independently, functions and constants, and a distinguished element of $@ \in W$, which acts as the “actual world”. The entire domain D is available at every world.

Our interpretation function I , then has two main roles: to correctly pick out the correct element of the domain given a term, and to evaluate atomic sentences at any given world. The latter role depends on the former. The former requires we give an interpretation to the constants and functions in our vocabulary. So, if $c \in \tau$ is a constant, we require that $I(c)$ is defined, and maps to some element of D . Also, if $f \in \tau$ is an n -ary function, we require that $I(f)$ is defined, and maps to some n -ary function on D . Then, given a term t of $\text{LPQ} \Rightarrow(\tau)$, I can inductively return the corresponding element of D . Note: all of this is world-independent.

In addition to supplying this information for terms and constants, we must then define, world-relatively, the extension and anti-extension of each predicate $P \in \tau$. I.e., $I(w, P) = I_w(P) = \langle P^+, P^- \rangle$, where $P^+ \cup P^- = D^n$ (where n is the arity of P), and P^+ denotes the set of n -tuples in D of which P holds, and

⁴A set containing our function, relation, and constant symbols.

⁵This represents the language when we add a name for each element of the domain to τ – we will write $d \in D$ for the name of the object d .

P^- denotes the set of n -tuples in D of which P is false. P is classical, then if and only if $P^+ \cap P^- = \emptyset$. P^+ is the extension of P , and P^- is the anti-extension of P .

Then, given an atomic sentence $P^n t_1 \dots t_n$ of $\text{LPQ} \Rightarrow (\tau^D)$, where P^n is an n -ary predicate and $t_1, \dots, t_n$ are terms of $\text{LPQ} \Rightarrow (\tau^D)$, we can define $I(w, P^n t_1 \dots t_n) = I_w(P^n t_1 \dots t_n)$ in the following way: we write $\langle P^+, P^- \rangle = I_w(P^n)$; and say $1 \in I_w(P^n t_1 \dots t_n)$ if and only if $\langle I(t_1), \dots, I(t_n) \rangle \in P^+$; and $0 \in I_w(P^n t_1 \dots t_n)$ if and only if $\langle I(t_1), \dots, I(t_n) \rangle \in P^-$. This lets I fulfil its latter role.

We require R have the following three restrictions: first, for all $w \in W$, $@Rw$. Second, a “semi-irreflexivity” condition: wRw for $w \in W$ implies that $w = @$. Third, for all $w, w' \in W$, we require that for any atomic sentence ϕ of $\text{LPQ} \Rightarrow (\tau^D)$ that wRw' implies that $I_w(\phi) \subseteq I_{w'}(\phi)$.

Given a τ -structure, we define a valuation function ν , which takes a world and a sentence, and maps it to a non-empty subset of $\{0, 1\}$. We often write $\nu_w(\phi)$ instead of $\nu(w, \phi)$.

We can define ν recursively as follows for $\mathfrak{M} = \langle W, D, R, I, @ \rangle$:

- if ϕ is atomic, then for all $w \in W$, $\nu_w(\phi) = I_w(\phi)$;
- if $\phi = (\psi \wedge \chi)$, then for all $w \in W$, $1 \in \nu_w(\phi)$ if and only if $1 \in \nu_w(\psi)$ and $1 \in \nu_w(\chi)$; and $0 \in \nu_w(\phi)$ if and only if $0 \in \nu_w(\psi)$ or $0 \in \nu_w(\chi)$;
- if $\phi = \neg\psi$, then for all $w \in W$, $1 \in \nu_w(\phi)$ if and only if $0 \in \nu_w(\psi)$; and $0 \in \nu_w(\phi)$ if and only if $1 \in \nu_w(\psi)$;
- if $\phi = \forall x\psi(x)$, then for all $w \in W$, $1 \in \nu_w(\phi)$ if and only if for all $d \in D$, $1 \in \nu_w(\psi(d))$; and $0 \in \nu_w(\phi)$ if and only if for some $d \in D$, $0 \in \nu_w(\psi(d))$; and
- if $\phi = (\psi \Rightarrow \chi)$, then for all $w \in W$, $1 \in \nu_w(\phi)$ if and only if for all $w' \in W$ such that wRw' , $1 \in \nu_{w'}(\psi)$ implies that $1 \in \nu_{w'}(\chi)$; and $0 \in \nu_w(\phi)$ if and only if for all $w' \in W$ such that wRw' , $1 \in \nu_{w'}(\psi)$ implies that $0 \in \nu_{w'}(\chi)$.

We shall now define logical consequence for $\text{LPQ} \Rightarrow$. Given Γ a set of sentences of $\text{LPQ} \Rightarrow (\tau)$, and a sentence $\phi \in \text{LPQ} \Rightarrow (\tau)$, we write $\Gamma \models \phi$ if and only if for every τ -structure $\langle W, D, R, I, @ \rangle$ where for the corresponding valuation function, ν , $1 \in \nu_{@}(\gamma)$ for each $\gamma \in \Gamma$, we also have that $1 \in \nu_{@}(\phi)$.

In addition to $\Rightarrow$, we can define a conditional $\rightarrow$, which contraposes, as $(\phi \rightarrow \psi) := (\phi \Rightarrow \psi) \wedge (\neg\psi \Rightarrow \neg\phi)$. In practice, $\rightarrow$ will be our conditional of choice, but, if for some reason it turns out troublesome, we can always switch to $\Rightarrow$ and give up contraposition – the reader can provide a simple counter-model. The semantics for this conditional are worked out in full here for an easier time writing (and reading) proofs:

- $1 \in \nu_w(\phi \rightarrow \psi)$ if and only if for all $w' \in W$ such that wRw' , $1 \in \nu_{w'}(\phi)$ implies that $1 \in \nu_{w'}(\psi)$; and $0 \in \nu_w(\phi \rightarrow \psi)$ implies that $0 \in \nu_{w'}(\phi)$.
- $0 \in \nu_w(\phi \rightarrow \psi)$ if and only if either for all $w' \in W$ such that wRw' , $1 \in \nu_{w'}(\phi)$ implies that $0 \in \nu_{w'}(\psi)$; or for all $w'' \in W$ such that wRw'' , $0 \in \nu_{w''}(\psi)$ implies that $1 \in \nu_{w''}(\phi)$.

We define the bi-conditionals: $(\phi \leftrightarrow \psi) := ((\phi \rightarrow \psi) \wedge (\psi \rightarrow \phi))$, $(\phi \Leftrightarrow \psi) := ((\phi \Rightarrow \psi) \wedge (\psi \Rightarrow \phi))$.

3.2 Second-Order Semantics

We extend $\text{LPQ} \Rightarrow$, adding semantics for second-order quantification, calling the resulting logic $\text{LPQ}^2 \Rightarrow$, and writing $\text{LPQ}^2 \Rightarrow (\tau)$ for the set of $\text{LPQ}^2 \Rightarrow$ -formulae.

The atomic formulæ of $\text{LPQ}^2 \Rightarrow (\tau)$ are the smallest set satisfying:

- the atomic formulæ of $\text{LPQ} \Rightarrow (\tau)$; and
- if X^n is an n -ary second-order variable, and $t_1, \dots, t_n$ are $\text{LPQ} \Rightarrow (\tau)$ -terms, then $X^n t_1 \dots t_n$ is an atomic formula of $\text{LPQ}^2 \Rightarrow (\tau)$.

So $\text{LPQ}^2 \Rightarrow (\tau)$ is the smallest set satisfying:

- the atomic formulæ of $\text{LPQ}^2 \Rightarrow (\tau)$ are all in $\text{LPQ}^2 \Rightarrow (\tau)$; and
- if $\phi, \psi \in \text{LPQ}^2 \Rightarrow (\tau)$, x is a first-order variable, and X is a second-order variable, then $(\phi \wedge \psi)$, $\neg \phi$, $(\phi \Rightarrow \psi)$, $\forall x \phi$, and $\forall X \phi$ are in $\text{LPQ}^2 \Rightarrow (\tau)$.

Then, $\exists X \phi := \neg \forall X \neg \phi$. And, we define the concept of being bound for second-order variables analogously to first-order variable; and a second-order variable that is not bound is free. A sentence of $\text{LPQ}^2 \Rightarrow (\tau)$ is any $\phi \in \text{LPQ}^2 \Rightarrow (\tau)$ with no free variables (whether they be first or second-order).

Given a vocabulary τ , we define a τ -structure of $\text{LPQ}^2 \Rightarrow$ as $\langle W, D, R, I, @ \rangle$, just as in $\text{LPQ} \Rightarrow$. We impose the same conditions on $W, D, R, I, @$, and logical consequence, but relativised to $\text{LPQ}^2 \Rightarrow$. In addition, however, I is defined to take an atomic sentence from $\text{LPQ}^2 \Rightarrow (\tau^{D,P})$, where $\tau^{D,P}$ is defined to extend τ^D with a name for each possible n -place predicate⁶ defined on D . Moreover, if Q is a predicate in $\tau^{D,P}$, but not in τ , then we require that for all $w, w' \in W$, $I_w(Q) = I_{w'}(Q)$ – and, we require that for each possible n -place paraconsistent (including consistent) predicate Q^n ⁷, there is a name K^n for a predicate in $\text{LPQ}^2 \Rightarrow (\tau^{D,P})$ such that $I(K^n) = Q^n$.

ν is as for $\text{LPQ} \Rightarrow$, but with a new clause for the second-order quantifier case:

- $1 \in \nu_w(\forall X^n \phi(X^n))$ if and only if every n -place predicate Q^n of $\text{LPQ}^2 \Rightarrow (\tau^{D,P})$ is such that $1 \in \nu_w(\phi(Q^n))$.
- $0 \in \nu_w(\forall X^n \phi(X^n))$ if and only if some n -place predicate Q^n of $\text{LPQ}^2 \Rightarrow (\tau^{D,P})$ is such that $0 \in \nu_w(\phi(Q^n))$.

That is, we can quantify over all possible extension anti-extension pairs, so long as every n -tuple for n the arity of the predicate, appears in at least the extension or the anti-extension. We see, in section 4, that this is problematic, and restrict ourselves to quantifying over fewer predicates – defining a new logic, which will have the same properties discussed in the next subsection as $\text{LPQ}^2 \Rightarrow$.

3.3 Some Properties of $\text{LPQ}^2 \Rightarrow$

Here, we see some properties of $\text{LPQ}^2 \Rightarrow (\tau)$. The first is a general property that gives good pragmatic reason to work with the logic. Roughly, the truth values always filter forward into any world accessible from any given world.

The second results are about contraction. We will show that the conditionals $\Rightarrow$ and $\rightarrow$ of $\text{LPQ}^2 \Rightarrow$ are $(n+1)/n$ contraction free. This is not enough to guarantee that the logic is able to support Naïve Set Theory however. Restall [Res93] has conjectured that a logic is able to support Naïve Set Theory if and only if it is robustly contraction free. We say that a logic is robustly contraction free if and only if there is no operator $>$ such that $A \rightarrow B \models A > B$, $A, A > B \models B$, and $A > (A > B) \models A > B$, for our conditional of choice $\rightarrow$. And Restall proves, in the same paper, that if a logic is not robustly contraction free, it cannot support Naïve Set Theory (without the model of Naïve Set

⁶In the paraconsistent sense: a pair with an extension and anti-extension, covering all of D^n .

⁷I.e., extension, anti-extension pair covering D^n .

Theory being trivial). We will not be able to prove that result here, however, but we are able to show that the conditionals are $(n + 1)/n$ contraction free.

3.3.1 Truth/Falsity of Sentences Filters Forward

Let us write γ^* to denote the application of the function $\cdot^* : \{0, 1\} \rightarrow \{0, 1\}$ which is defined to be the only non-identity permutation of $\{0, 1\}$ (i.e., classical negation).

Theorem 3.3.1. *Let $\mathfrak{M} := \langle W, D, R, I, @ \rangle$ be a τ -structure, $\phi \in \text{LPQ}^2 \Rightarrow (\tau^{D,P})$ a sentence, and ν the valuation function corresponding to $\mathfrak{M}$. Then, for any $w \in W$, $x \in \{0, 1\}$, and $w' \in W$ such that wRw' , $x \in \nu_w(\phi)$ implies that $x \in \nu_{w'}(\phi)$.*

Proof: we shall proceed by induction on the complexity of ϕ . Our basis case, when ϕ is atomic, holds by definition. We have different cases to consider for our induction step:

Negation If $\phi = \neg\psi$, then $x \in \nu_w(\phi)$ implies that $x^* \in \nu_w(\psi)$, and, so, by our induction hypothesis, $x^* \in \nu_{w'}(\psi)$ for all w' such that wRw' , and so, $x \in \nu_{w'}(\phi)$.

Conjunction If $\phi = (\psi \wedge \chi)$, then $1 \in \nu_w(\phi)$ implies that $1 \in \nu_w(\psi)$ and $1 \in \nu_w(\chi)$, and, so, by our induction hypothesis, $1 \in \nu_{w'}(\psi)$ and $1 \in \nu_{w'}(\chi)$, for all w' such that wRw' , and, so, $1 \in \nu_{w'}(\phi)$. Alternatively, if $0 \in \nu_w(\phi)$, then, $0 \in \nu_w(\psi)$ or $0 \in \nu_w(\chi)$, and, so, by our induction hypothesis, $0 \in \nu_{w'}(\psi)$ for all w' such that wRw' or $0 \in \nu_{w'}(\chi)$ for all w' such that wRw' ; and, so, $0 \in \nu_{w'}(\phi)$ for all w' such that wRw' .

Implication If $\phi = (\psi \Rightarrow \chi)$, then $1 \in \nu_w(\phi)$ implies that for all w' such that wRw' , $1 \in \nu_{w'}(\chi)$ if $1 \in \nu_{w'}(\psi)$, and, so, by the fact that R is transitive, for all w'' such that $w'Rw''$, wRw'' , a fortiori $1 \in \nu_{w''}(\chi)$ if $1 \in \nu_{w''}(\psi)$; and, so $1 \in \nu_{w'}(\phi)$. Alternatively, if $0 \in \nu_w(\phi)$, then for all w' such that wRw' , $0 \in \nu_{w'}(\chi)$ if $1 \in \nu_{w'}(\psi)$. So, if w'' is such that $w'Rw''$, wRw'' as R is transitive, and $0 \in \nu_{w''}(\chi)$ if $1 \in \nu_{w''}(\psi)$. Thus, $0 \in \nu_{w'}(\phi)$.

First-Order Universal Quantification If $\phi = \forall x\psi(x)$, then $1 \in \nu_w(\phi)$ implies that for all $d \in D$, $1 \in \nu_w(\psi(d))$, and, so, by our induction hypothesis, $1 \in \nu_{w'}(\psi(d))$ for all w' such that wRw' , and so $1 \in \nu_{w'}(\phi)$. Alternatively, if $0 \in \nu_w(\phi)$, then for some $d \in D$, $0 \in \nu_w(\psi(d))$, and, so, by our induction hypothesis, $0 \in \nu_{w'}(\psi(d))$ for all w' such that wRw' ; and, so $0 \in \nu_{w'}(\phi)$.

Second-Order Universal Quantification If $\phi = \forall X^n\psi(X^n)$, then $1 \in \nu_w(\phi)$ implies that for any P with $I_w(P) = \langle P^+, P^- \rangle$ and $P^+ \cup P^- = D^n$, we must have that $1 \in \nu_w(\psi(P))$, and so, by our induction hypothesis, $1 \in \nu_{w'}(\psi(P))$ for any $w' \in W$ with wRw' ; and so $1 \in \nu_{w'}(\phi)$. Alternatively, if $0 \in \nu_w(\phi)$, then for some P with $I_w(P) = \langle P^+, P^- \rangle$ and $P^+ \cup P^- = D^n$, we must have that $0 \in \nu_w(\psi(P))$, and so, by our induction hypothesis, $0 \in \nu_{w'}(\psi(P))$ for any $w' \in W$ with wRw' ; and so, $0 \in \nu_{w'}(\phi)$. ■

We can use this to show that for any sentence $\phi \in \text{LPQ}^2 \Rightarrow (\tau)$, for all $w \in W$, $\nu_w(\phi) \neq \emptyset$, and $\nu_w(\phi) \subseteq \{0, 1\}$, both of which we require as this is a solely paraconsistent (opposed to paracomplete) logic. In particular, the only case we need to verify is the conditional, and all of the other cases follow (by case, I mean if we were to induct on the complexity of ϕ) by the fact they hold for LPQ, and it is obvious for the second-order quantification case. To see it holds for conditionals, note, obviously, if $\phi = (\psi \Rightarrow \chi)$, then for all $w \in W$, $\nu_w(\phi) \subseteq \{0, 1\}$, by definition. Then, to see $\nu_w(\phi) \neq \emptyset$, assume that it is: so $0 \notin \nu_w(\phi)$ and $1 \notin \nu_w(\phi)$. Then by our above conditions, there are $w', w'' \in W$ such that wRw' and wRw'' with $1 \in \nu_{w'}(\psi)$ and $1 \notin \nu_{w'}(\chi)$, and so, $1 \notin \nu_w(\chi)$ for otherwise $1 \in \nu_{w'}(\chi)$ by Theorem 3.3.1. But, w'' is such that $1 \in \nu_{w''}(\psi)$ and $0 \notin \nu_{w''}(\chi)$, and so, $0 \notin \nu_w(\chi)$, by the same

reasoning as before. But, then, this violates our induction hypothesis that $\emptyset \neq \nu_w(\chi) \subseteq \{0, 1\}$. Hence, either $0 \in \nu_w(\phi)$ or $1 \in \nu_w(\phi)$.

As we have just seen, this (Theorem 3.3.1) is a nice result, and makes proofs easy to work with in this logic. And gives us some leeway to philosophically justify the conditional semantics: we examine ways the model could “go wrong” (in terms of ways it could be more consistent), and see if the conditional would still hold in these situations.

3.3.2 Conditionals are $(n + 1)/n$ Contraction Free

We shall now define what it means for an operator to be $(n + 1)/n$ contraction free. But, first we need to define some sentences. We can inductively define $\phi_n^>$ for an operator $>$ as follows: $\phi_0^> := A_2$ and $\phi_{n+1}^> := (A_1 > \phi_n^>)$.

We say an operator $>$ is $(n + 1)/n$ contraction free if for all positive integral n , $\phi_{n+1}^> \neq \phi_n^>$. Restall [Res94, Chapter 13] showed if we have an operator $>$ which is not $(n + 1)/n$ contraction free, satisfies *modus ponens*, and is derivable from a conditional of choice, then the logic is not robustly contraction free. These results that the conditionals of $\text{LPQ}^2 \Rightarrow$ are $(n + 1)/n$ contraction free shows the issue (if there at all) does not lie with the conditionals themselves. This provides some⁸ assurance $\text{LPQ}^2 \Rightarrow$ is itself $(n + 1)/n$ contraction free.

We will now define our counter-models, and then we shall show that these are indeed counter-models, for each positive integral n to $\phi_{n+1}^{\rightarrow} \models \phi_n^{\rightarrow}$. $\mathfrak{M}_n$ is a $\{A_1, \dots, A_n, c\}$ -structure, where each A_i is a 1-place predicate and c a name. We will abuse notation here, and write A_i as shorthand for $A_i c$. $\mathfrak{M}_n := \langle W, D, R, I, @ \rangle$, where we define each element as follows.

$W := \{w_1, w_2, \dots, w_{2^n}\}$. $D := \{d\}$, and the name c of our vocabulary refers to d . $@ := w_1$. R is defined in such a way that given distinct $v, w \in W$, vRw if and only if for each A_i , $\nu_v(A_i) \subseteq \nu_w(A_i)$. Moreover, $@R@$, and wRw for $w \in W$ implies that $@R@$. Clearly, then, R is transitive, and satisfies our feeding-forward of atomic formulæ – as our only atomic formulæ are A_i s. Finally, $I_{w_i}(A_j) = \{0, 1\}$ (and $I_{w_i}(A_j) = \{0\}$ otherwise) if and only if when we write $i - 1$ in binary, with the least-significant bit on the left, and ensuring that there are n digits⁹ – adding extra 0s if required, the j th digit (from the left, and counting from 1) is a 1.

Theorem 3.3.2. $\text{LPQ}^2 \Rightarrow$ is such that for any positive integral n , $\phi_{n+1}^{\rightarrow} \neq \phi_n^{\rightarrow}$.

Proof: we proceed by induction on n . Our basis case is when $n = 1$. A counter-model is $\mathfrak{M}_2$.

Our induction hypothesis is that for all $n \leq k$, then $\mathfrak{M}_{n+1}$ is such that

- the world $w \in W$ where $\nu_w(A_1) = \{0, 1\}$, and for $1 < i \leq n + 1$, $\nu_w(A_i) = \{0\}$ is such that $\nu_w(\phi_{n-1}^{\rightarrow}) = \{0\}$;
- $\nu_{@}(\phi_n^{\rightarrow}) = \{0\}$;
- $1 \in \nu_{@}(\phi_{n+1}^{\rightarrow})$.

We will now verify that $\mathfrak{M}_2$ satisfies these three items. For the first, $\phi_{n-1}^{\rightarrow}$, when $n = 1$ is $\phi_0^{\rightarrow} := A_2$, so we require that in the world where only A_1 is satisfied, A_2 is not. Well, this holds by definition (and such a world clearly exists). For the second item, $\phi_n^{\rightarrow}$ is $A_1 \rightarrow A_2$, we can see this is easily not satisfied

⁸Although, admittedly, not much.

⁹I.e., if $n = 3$, and $i = 4$, then $i - 1$ is written as 110.

in virtue of the world discussed previously: where only A_1 is satisfied but A_2 is not. Finally, for the third item, $\phi_{n+1}^{\rightarrow}$ is $A_1 \rightarrow (A_1 \rightarrow A_2)$. As $0 \in \nu_{@}(A_1 \rightarrow A_2) \cap \nu_{@}(A_1)$, we know that in every world the “contraposing” clause of our conditional is satisfied: for all $w \in \mathcal{W}$, $0 \in \nu_w(A_1 \rightarrow A_2)$ implies that $0 \in \nu_w(A_1)$. So, we just need to show that for all $w \in \mathcal{W}$, $1 \in \nu_w(A_1)$ implies that $1 \in \nu_w(A_1 \rightarrow A_2)$. This follows if $1 \in \nu_w(A_1 \rightarrow A_2)$ is true for our world w in our proof of the first item. This is certainly true, because the only world that is accessible from w is the trivial world. Hence, item three is proved.

Induction step: $n = k + 1$. First, note that by our induction hypothesis, there is a world w in $\mathfrak{M}_{k+1}$ such that $\nu_w(A_1) = \{0, 1\}$ and for $1 < i \leq k + 1$, $\nu_w(A_i) = \{0\}$, and $\nu_w(\phi_{k-1}^{\rightarrow}) = \{0\}$.

Then, it is obvious that the world w' in $\mathfrak{M}_{k+2}$ such that $\nu_{w'}(A_1) = \{0, 1\} = \nu_{w'}(A_{k+2})$ and for $1 < i \leq k + 1$, $\nu_{w'}(A_i) = \{0\}$ is such that $\nu_{w'}(\phi_{k-1}^{\rightarrow}) = \{0\}$. Because w' , and the worlds accessible from w' are identical to w and those accessible from w ; i.e., same number, relations, truth values; except with respect to A_{k+2} , which does not even exist in w and the worlds accessible from w , and A_{k+2} is a glut in w' and the worlds accessible from w' , and clearly A_{k+2} being glutty has nothing to do with the conditional facts about A_1 and A_2 – so we see that w and w' are “isomorphic” with respect to A_1 and A_2 , and consequently the conditional facts about A_1 and A_2 .

But, then, the world w'' in $\mathfrak{M}_{k+2}$ such that $\nu_{w''}(A_1) = \{0, 1\}$ and for $1 < i \leq k + 2$, $\nu_{w''}(A_i) = \{0\}$ is such that $\nu_{w''}(\phi_k^{\rightarrow}) = \{0\}$ because $w''Rw'$ and $1 \in \nu_{w'}(A_1)$, but $1 \notin \nu_{w'}(\phi_{k-1}^{\rightarrow}) = \{0\}$. This satisfies the first item of our induction.

The second item, then follows because $@$ of $\mathfrak{M}_{k+2}$ is such that $@Rw''$, and $1 \in \nu_{w''}(A_1)$, but $1 \notin \nu_{w''}(\phi_k^{\rightarrow})$.

Finally, note that $1 \in \nu_{@}(A_1)$ implies that $1 \in \nu_{@}(\phi_{k+1}^{\rightarrow})$ vacuously, and $0 \in \nu_{@}(\phi_{k+1}^{\rightarrow})$ implies that $0 \in \nu_{@}(A_1)$ because $0 \in \nu_{@}(A_1)$ by construction. Moreover, as $0 \in \nu_{@}(\phi_{k+1}^{\rightarrow})$ and $0 \in \nu_{@}(A_1)$, for all worlds w of $\mathfrak{M}_{k+2}$, $0 \in \nu_w(\phi_{k+1}^{\rightarrow})$ and $0 \in \nu_w(A_1)$; so, clearly, for all w such that $wR@$, $0 \in \nu_w(A_1)$ implies that $0 \in \nu_w(\phi_{k+1}^{\rightarrow})$. Then the result follows if for all worlds w of $\mathfrak{M}_{k+2}$, $1 \in \nu_w(A_1)$ implies $1 \in \nu_w(\phi_{k+1}^{\rightarrow})$.

Note this follows if $1 \in \nu_{w''}(\phi_{k+1}^{\rightarrow})$, as this is the “first” world where A_1 is true. This is the case. For, in every world w accessible from w'' , we must have that $\nu_w(A_1) = \{0, 1\}$, and so in order to show that a sentence of the form $\phi_l^{\rightarrow}$ for some k holds at each w , we just need to show that $\phi_{l-1}^{\rightarrow}$ holds at every world accessible from w .

We can define a concept called “layer”. We say the i th layer of worlds accessible from w'' is defined inductively as follows: the 0th layer contains only w'' ¹⁰ Then, the $(n + 1)$ th layer is a subset of the worlds of $\mathfrak{M}_{k+2}$ such that each w in the $(n + 1)$ th layer is such that $w''Rw$, and, the $(n + 1)$ th layer contains all and only those worlds w which are such that there is a world v in the n th layer with vRw and there does not exist any world u such that vRu and uRw – i.e., there is no world between the two.

By our earlier observation, every world in one of these layers has a glut at A_1 .

We can see there are $k + 2$ layers¹¹ and the $(k + 1)$ th layer contains only the fully trivial world. Then, in the $(k + 1)$ th layer, every world w is such that $1 \in \nu_w(A_1) \cap \nu_w(A_2)$ as it is fully trivial (or we can note that the conditional is trivially satisfied by the lack of a world accessible from w). So, every world w in

¹⁰ w'' is not accessible from w'' , hence we 0-index it.

¹¹ The i th layer is such that the number of gluts, apart from A_1 , in each of the worlds it contains is equal to i – each layer can be characterised totally by this property: it contains only and all those worlds with A_1 glutty and i other A_j s glutty. There are $k + 1$ other A_j s, so $k + 2$ layers.

the k th layer is such that $1 \in \nu_w(\phi_1^\rightarrow)$ as well as $1 \in \nu_w(A_1)$. Moreover, every world w in the $(k-1)$ th layer is such that $1 \in \nu_w(\phi_2^\rightarrow)$, as well as $1 \in \nu_w(A_1)$. This continues until the 0th layer, and, so, we inductively get that $1 \in \nu_{w''}(\phi_{k+1}^\rightarrow)$, completing the proof. ■

Corollary 3.3.3. $\text{LPQ}^2 \Rightarrow$ is such that for any positive integral n , $\phi_{n+1}^\rightarrow \neq \phi_n^\rightarrow$.

Proof: it can be easily verified that for any formulae ϕ, ψ , we must have $\phi \rightarrow \psi \models \phi \Rightarrow \psi$, and so, we see that $\mathfrak{M}_{n+1}$ models $\phi_{n+1}^\rightarrow$ as it models $\phi_{n+1}^\rightarrow$, by Theorem 3.3.2. Similarly, by the proof of Theorem 3.3.2, we see that $\mathfrak{M}_{n+1}$ does not model $\phi_n^\rightarrow$ in virtue of the fact that it does not model $\phi_n^\rightarrow$. *a fortiori*, $\mathfrak{M}_{n+1}$ does not model $\phi_n^\rightarrow$, and so $\mathfrak{M}_{n+1}$ is a suitable counter-model. ■

Corollary 3.3.4. $\text{LPQ}^2 \Rightarrow$ is such that for any positive integral n , $\phi_{n+1}^\rightarrow \neq \phi_n^\rightarrow$.

Proof: if $\phi_{n+1}^\rightarrow \models \phi_n^\rightarrow$, then $\phi_{n+1}^\rightarrow \models \phi_n^\rightarrow$, violating Corollary 3.3.3. ■

Corollary 3.3.5. $\text{LPQ}^2 \Rightarrow$ is such that for any positive integral n , $\phi_{n+1}^\rightarrow \neq \phi_n^\rightarrow$.

Proof: the proof is similar to that of Corollary 3.3.3, and $\mathfrak{M}_{n+1}$ is a suitable counter-model. ■

These results are nice, but do not guarantee that the logic is robustly contraction free, and work in this area should be a priority if one wants to use $\text{LPQ}^2 \Rightarrow$ for Candid (or even Naïve) Set Theory – or, in particular its $(\text{LPQ}^2 \Rightarrow)$'s derivatives which we shall see later on.

We now see $\text{LPQ}^2 \Rightarrow$ is not suitable for the job. We shall see a problem with our second-order quantification, and will need to restrict it accordingly.

4 Identity and Second-Order Comprehension

Usually, when one uses second-order logic, it is convenient to define identity using second-order quantification. For example, here is such a typical way:

$$(x = y) := \forall X^1 (X^1 x \leftrightarrow X^1 y).$$

The issue with this approach (which is similar to the reasoning in [HP18, pp. 5–6] – who suggest that in second-order LP, we ought to define identity as a primitive) in $\text{LPQ}^2 \Rightarrow$ is that when we quantify over the one-place predicates using $\forall X^1$, we quantify over the predicate Q^1 , which is such that for $w \in \mathcal{W}$, $I_w(Q^1) = \langle D, D \rangle$. I.e., it holds – and, indeed, does not hold – for any object of the domain. This, causes us to have that $0 \in \nu_w(x = y)$ for any objects $x, y \in D$. But this means that even for truly identical objects, the identity claim is evaluated as (at least) false. This certainly is problematic if we are taking our semantics seriously (which I hope to do).

What I suggest, (as suggested in [HP18] we could instead define identity as a primitive) is to modify our logic so we can only quantify over the predicates in the vocabulary, and all the possible classical predicates. In fact, the problem with identity is quite general: every second-order universal quantifier

$\forall X\phi(X)$, where X is genuinely free in ϕ , will, on these semantics turn out to be (at least) false – and sentences of the form $\exists X\phi(X)$, with X genuinely free in ϕ , will turn out to be (at least) true. So, I think this also provides further evidence for changing the logic. Let us see, then, what such a new logic would look like; I will call it $\text{LPQ}^{2c} \Rightarrow$ to represent our restriction to only classical non-vocabulary predicates.

4.1 $\text{LPQ}^{2c} \Rightarrow$ Semantics

The only change, then, that we make to $\text{LPQ}^2 \Rightarrow$, is that we restrict second-order quantification to classical predicates, and those of the vocabulary. Our semantics are identical otherwise: the same terms, formulæ, $\mathcal{E}^c$. A *prima facie* justification for this change in semantics is that we do not want to postulate more gluts than necessary: by only quantifying over classical predicates (that are not in the vocabulary), we are not pre-supposing any gluts. If there are any glutty predicates, then they come from our vocabulary and our interpretations in the models.

In order to restrict our attention to classical predicates and those already in the vocabulary, we define τ^{D, P^c} to only add names for classical predicates – opposed to $\tau^{D, P}$ which adds names for all paraconsistent predicates. Then, we ensure that if $Q^n \in \tau^{D, P^c}$ is a predicate but $Q^n \notin \tau$, then for all w , $I_w(Q^n) = \langle Q^+, Q^- \rangle$, where $Q^+ \cap Q^- = \emptyset$, ensuring that it is classical. And we still have the condition that for all $w, w' \in W$, $I_w(Q^n) = I_{w'}(Q^n)$. Then our clauses for ν use $\text{LPQ}^{2c} \Rightarrow (\tau^{D, P^c})$ instead of $\text{LPQ}^2 \Rightarrow (\tau^{D, P})$:

- $1 \in \nu_w(\forall X^n \phi(X^n))$ if and only if every n -place predicate Q^n of $\text{LPQ}^{2c} \Rightarrow (\tau^{D, P^c})$ is such that $1 \in \nu_w(\phi(Q^n))$.
- $0 \in \nu_w(\forall X^n \phi(X^n))$ if and only if some n -place predicate Q^n of $\text{LPQ}^{2c} \Rightarrow (\tau^{D, P^c})$ is such that $0 \in \nu_w(\phi(Q^n))$.

Note, that all our results from subsection 3.3 still hold for $\text{LPQ}^{2c} \Rightarrow$.

4.2 Identity Re-visited

Let us now consider how identity works in $\text{LPQ}^{2c} \Rightarrow$. Defining it in the following way:

$$(x = y) := \forall X^1 (X^1 x \leftrightarrow X^1 y).$$

First, consider d_1 of the domain. Then, certainly, for all $w \in W$, $1 \in \nu_w(d_1 = d_1)$ on the above definition, as d_1 is in the same extensions and anti-extensions as itself by definition. If we consider d_1 and d_2 of the domain, which are distinct elements of D , then there there is a classical predicate which is (only) true of one, and (only) false of the other, so $0 \in \nu_w(d_1 = d_2)$.

How, then, do we get glutty identity claims? First, we shall observe that if $1 \in \nu_w(d_1 = d_2)$, then $d_1 = d_2$ in the meta-theory; i.e., they are the same element of D . To see this: if $1 \in \nu_w(d_1 = d_2)$ means that $1 \in \nu_w(X^1 d_1 \leftrightarrow X^1 d_2)$ for every classical X^1 , and as $\nu_w(X^1 d) = \nu_{w'}(X^1 d)$ for any worlds w, w' , by our definition, we must have that $Y d_2$ holds where Y is the classical predicate that is only true of d_1 and nothing else. This ensures $d_1 = d_2$ in the meta-theory. So, an identity claim $d_1 = d_2$, of the logic, is glutty if and only if $0 \in \nu_w(d_1 = d_2)$ and $d_1 = d_2$. To get $0 \in \nu_w(d_1 = d_2)$ in this case, we need a one-place predicate of the vocabulary which has a glut at d_1 (or at d_2 as they are equal in the meta-theory). So, we only have gluts in cases where there is genuine identity (from the perspective of the meta-theory) and a glut occurs, at the point in question, for some predicate of the vocabulary.

This, to me, seems like a nice characterisation of paraconsistent identity. And, I would be particularly interested to see how it plays out in something like inconsistent arithmetic.

4.3 Second-Order Comprehension

The issue now, however, is that we cannot rely on second-order comprehension, which is also commonly used in classical second-order logic. Second-order comprehension is the the axiom schema, such that for any ϕ with n free variables, we have

$$\exists X^n \forall x_1 \dots \forall x_n [X^n x_1 \dots x_n \leftrightarrow \phi(x_1, \dots, x_n)].$$

I.e., there is an n -place predicate, for any formula with n free variables such that they agree (from the perspective of the logic) on truth and falsity. This means that we can quantify over first-order formulæ in some sense (although not second-order).

The reason why this does not work in $\text{LPQ}^{2c} \Rightarrow$ is that we only can quantify over our predicates of the vocabulary (which all have fixed arity), and the classical predicates. The classical predicates cannot encode inconsistent information as they are classical, so can't be used to encode information about inconsistent sentences – which could always possibly exist. And the predicates of the language are already being used.

Why this is a problem shall be explained in our next section, as it makes some things awkward for us when we return to Naïve Set Theory.

5 Naïve Comprehension

Recall, then, the Axiom Schema of Naïve Comprehension, where $\phi(x)$ is a formula with all free variables among x :

$$\exists y \forall x [x \in y \leftrightarrow \phi(x)].$$

If $\text{LPQ}^{2c} \Rightarrow$ satisfied the second-order comprehension schema, we would be able to define the Axiom Schema of Naïve Comprehension as

$$\forall X^1 \exists y \forall x [x \in y \leftrightarrow X^1 x],$$

noting that it would no longer be a schema, but a single axiom.

We do not have that option available to us. And, so, if we are to use the power of second-order logic¹², we need to rethink our approach. What I suggest, then, is that we return to a classical compositionality approach. That is to say, we indeed have our Axiom of Naïve Comprehension (in $\text{LPQ}^{2c} \Rightarrow$), which lets us use all sets defined by any arbitrary one-place classical predicate, in addition to the sets corresponding to the one-place predicates of our vocabulary¹³.

¹²Certainly, we could still just add the Axiom Schema of Naïve Comprehension. I am just running with the idea that we should utilise second-order logic; and, at this point, the reader might be discouraged from using second-order logic in Naïve Set Theory. I think this is a valid response.

¹³It would be beneficial to ensure that the model has interpretations and names for all the “restricted” versions of predicates in the language: that is, for $n > 1$, if P^n is an n -place predicate, then for each $\bar{x} \in D^{n-1}$, there is a predicate P_x^n , which is a one-place predicate defined to be $P^n \bar{x} y$, with y free – and, in addition, it would be beneficial to ensure that y could appear in any position of the predicate. Otherwise, we would need an axiom schema allowing all atomic formulæ of our logic to have a corresponding set. – This is a valid criticism: we still have to deal with all these cases. But, I am just going “all in” on our Second-Order quantification in $\text{LPQ}^{2c} \Rightarrow$.

But, in this context, the Axiom of Naïve Comprehension does not entail the Axiom Schema of Naïve Comprehension, so we need to add extra axioms to do that work for us. I suggest we add axioms, much like the axiom of union to our formulation of Naïve Set Theory. I.e., we add an axiom – one for each operator of our logic: $\wedge$, $\neg$, $\forall x$, $\forall X$, and $\Rightarrow$ such that given the sets corresponding to $\phi(x)$ and $\psi(x)$, we can construct the sets corresponding to $\phi(x) \wedge \psi(x)$, $\neg\phi(x)$, $\forall x\phi(x)$, $\mathcal{E}_C$.

We can easily justify the first two. For conjunction, add an Axiom of Intersection:

$$\forall x\forall y\exists z\forall t [(t \in x \wedge t \in y) \leftrightarrow t \in z],$$

and so given $\{x; \phi(x)\}$ and $\{x; \psi(x)\}$, we can construct $\{x; \phi(x) \wedge \psi(x)\}$; the sets corresponding to $\phi(x)$ and $\psi(y)$ exist by a suitable induction hypothesis.

We can also add an Axiom of Complements:

$$\forall x\exists z\forall t [t \notin x \leftrightarrow t \in z],$$

which allows us to construct $\{x; \neg\phi(x)\}$ from $\{x; \phi(x)\}$. Which is also easy to justify naïvely (just look at naïve set theory in mathematical practice).

Now, what do we do for first and second order quantification and for the (non-contraposing) conditional? Quantification I shall, regrettably, leave for this essay.

For the conditional, then we want the following axiom:

$$\forall x\forall y\exists z\forall t [(t \in x \Rightarrow t \in y) \leftrightarrow t \in z],$$

allowing us to construct $\{x; \phi(x) \Rightarrow \psi(x)\}$ given $\{x; \phi(x)\}$ and $\{x; \psi(x)\}$. But, how would we justify such an axiom?

We revise our semantics again. The issue with justifying this axiom, for me, in $\text{LPQ}^{2c} \Rightarrow$, is as follows. My hope is that we use Candid Set Theory as our meta-theory, and reformulate Candid Set Theory inside our meta-theory. But, with our current modal semantics, it is impossible for us, from the perspective of the actual world in our model, to tell what other worlds there are. So, the truths here depend on things we cannot know, for our meta-theory is unable to tell as as our meta-theory depends on those facts – and so there is a vicious cycle.

Therefore, I believe we need to incorporate into $\text{LPQ}^{2c} \Rightarrow$ the (very strong) principle, that for any theory T , for each structure $\mathfrak{M}, \mathfrak{N}$ of our logic (which will be a sub-logic of $\text{LPQ}^{2c} \Rightarrow$) that validates T (i.e., is a model of T), if $I_{@ \mathfrak{M}} = I_{@ \mathfrak{N}}$ (noting that this also requires that $D^{\mathfrak{M}} = D^{\mathfrak{N}}$, and $\mathfrak{M}, \mathfrak{N}$ are structures of the same vocabulary), then $\mathfrak{M} = \mathfrak{N}$. Thus, that the entire structure is completely determined by how it treats the atomic formulæ in the actual world.

So, from the perspective of the actual world, we can deduce the entire structure. This justifies our use of the logic as our meta-theory, for we can, from the perspective of the actual world, deduce the structure of the entire model.¹⁴

Thus, this justifies our use of

$$\forall x\forall y\exists z\forall t [(t \in x \Rightarrow t \in y) \leftrightarrow t \in z],$$

because everything is already determined in the actual world.

Let us see the final version of our logic. I call it $\text{LPQ}^{2c} \Rightarrow^@$, for the modal part (coming from our conditional) depends only on the actual world.

¹⁴I use ‘we’ very loosely here – I mean something like “logic itself” as I am a realist; literal ‘we’ works for anti-realists.

5.1 A Final Semantics

Our logic, $\text{LPQ}^{2c} \Rightarrow^\circ$ is almost exactly the same as $\text{LPQ}^{2c} \Rightarrow$ but we “prune” some models, so that given two τ -structures, $\mathfrak{M}, \mathfrak{N}$, with the same domain D and satisfying the same theory T in question, are such that if they agree on the truth values (i.e., glutty iff glutty; only true iff only true; only false iff only false) of every atomic formula of $\text{LPQ}^{2c} \Rightarrow (\tau^{D, P^c})$ at the actual world, then $\mathfrak{M} = \mathfrak{N}$. The only difference is that in $\text{LPQ}^{2c} \Rightarrow^\circ$ we do not allow any worlds to agree on all the same atomic formulæ; i.e., for all $w, w' \in W$, $I_w = I_{w'}$ implies that $w = w'$.

We give a method to choose the “true model” from any number of candidates.

Assume we have $\mathcal{M}$ – a set of τ -structures $\mathfrak{M}$, all satisfying T , and all on the same domain D , but each $I_{@ \mathfrak{M}}$ is distinct from all the others. First, we calculate, for each $\mathfrak{M} \in \mathcal{M}$, $|W^{\mathfrak{M}}|$. We then discard or prune those models that do not have the maximal number of worlds.

This is the first principle: we prefer a maximal number of worlds. The reasoning for this is that it “pushes” the fully inconsistent world (which must appear if there is a maximal number) “further” into the model, and so is preferring consistency to inconsistency.

Then, if we still do not only have one model, we invoke our second principle: we prefer less truth – for we (at least I) see truth as requiring more of a burden than falsity to be the case. So, we count the number of atomic formulæ ϕ that are such that $1 \in I_{@ \mathfrak{M}}(\phi)$ for each $\mathfrak{M}$ still in $\mathcal{M}$, and we keep only those models with the least number of true atomic formulæ in the actual world.

Finally, if that is still not enough, we, unfortunately, invoke a somewhat arbitrary principle that settles the matter fully. Assuming the axiom of choice, well-order the atomic sentences of $\text{LPQ}^{2c} \Rightarrow (\tau^{D, P^c})$ as ϕ_α , where α is an ordinal. Then, repeat the following process, until we only have one element in $\mathcal{M}$, which does eventually terminate – for otherwise every model would be the same, which would only leave us with one model. Choose the models $\mathfrak{M}$ in $\mathcal{M}$ so that they differ from all the others with respect to $I_{@ \mathfrak{M}}(\phi_\beta)$, for the least β .

Note, the results of subsection 3.3 hold for $\text{LPQ}^{2c} \Rightarrow^\circ$, as the models used in Theorem 3.3.2 are the uniquely determined maximal models given the actual world; we do not need to consider the second or third principles.

6 Conclusion

I gave reasons for why I believe that Candid Set Theory should replace Naïve Set Theory. This took us on a path of exploring second-order paraconsistent logic. We saw a basic logic $\text{LPQ}^{2c} \Rightarrow$, which was not suitable for purpose; so, we moved to $\text{LPQ}^{2c} \Rightarrow^\circ$. Because I am sensitive to the idea that we should embed Candid Set Theory inside itself, I justified a final move to $\text{LPQ}^{2c} \Rightarrow^\circ$. We saw that the properties of subsection 3.3 still hold for $\text{LPQ}^{2c} \Rightarrow^\circ$, giving us some, albeit, not much hope that $\text{LPQ}^{2c} \Rightarrow^\circ$ can support Naïve Set Theory.

What we did not see, is that $\text{LPQ}^{2c} \Rightarrow^\circ$ is robustly contraction free, nor did we examine, in any detail, the inferences possible in $\text{LPQ}^{2c} \Rightarrow^\circ$. Moreover, we did not show that $\text{LPQ}^{2c} \Rightarrow^\circ$ could support Naïve Set Theory, and we did not prove that if $\text{LPQ}^{2c} \Rightarrow^\circ$ could support Naïve Set Theory, then we would have models of Candid Set Theory. I suggest these things for interested readers in the future. A possible roadmap for engagement with Candid Set Theory in $\text{LPQ}^{2c} \Rightarrow^\circ$ is:

- Check whether $\text{LPQ}^{2c} \Rightarrow^\circ$ has any totally objectionable inferences.

- Prove $\text{LPQ}^{2c} \Rightarrow^@$ is robustly contraction free.
- Prove $\text{LPQ}^{2c} \Rightarrow^@$ can support Naïve Set Theory.
- Prove $\text{LPQ}^{2c} \Rightarrow^@$ can support Candid Set Theory – and examine such models.

Any one of these steps could fail, and in which case the project, at least using $\text{LPQ}^{2c} \Rightarrow^@$, would have to be abandoned. The goal then would be to find another logic suitable for Candid Set Theory, which may not be second-order.

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